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Hiperciclicidad en espacios de funciones holomorfas y pseudo órbitas de operadores lineales.

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Hiperciclicidad en espacios de funciones holomorfas y pseudo órbitas de operadores lineales

Resumen

En esta tesis estudiamos distintos problemas sobre densidad de órbitas de operadores lineales. Un operador lineal se dice hipercíclico si admite una órbita densa. Podemos decir que el centro de atención es el comportamiento de las sucesivas iteraciones de un operador lineal. En otras palabras, se estudian sistemas dinámicos discretos asociados a operadores lineales. En el contexto finito dimensional este problema se puede resolver a través del estudio de la forma de Jordan asociada a una matriz, y los comportamientos son relativamente simples (de ahí que el caos se asocia naturalmente a sistemas no lineales). Sin embargo, en espacios de dimensión infinita los sistemas lineales pueden ser caóticos, ya que aparecen fenómenos nuevos, como por ejemplo la existencia de órbitas densas en todo el espacio.

Los primeros ejemplos de operadores hipercíclicos surgieron en el contexto de la teoría de funciones analíticas. Así, en 1929, G. D. Birkhoff [Bir29] probó que para todo $a \in \mathbb{C}$, $a \neq 0$, el operador traslación en el espacio de funciones enteras de variable compleja $(H(\mathbb{C}), \tau)$ con la topología compacto-abierta, $T_a : H(\mathbb{C}) \to H(\mathbb{C})$ definido por $T_a f(z) = f(z+a)$ es hipercíclico, y en 1952, G. R. MacLane [Mac52], demostró que lo mismo ocurre con el operador de diferenciación en $H(\mathbb{C})$. Estos resultados fueron generalizados por G. Godefroy y J. H. Shapiro en 1991 [GS91] quienes probaron que todo operador lineal y continuo $T : H(\mathbb{C}) \to H(\mathbb{C})$ que conmute con las traslaciones y no sea un múltiplo de la identidad es hipercíclico. Esta familia de operadores se conoce por el nombre de operadores de convolución. En esta tesis estudiamos operadores de convolución definidos en espacios de funciones holomorfas sobre espacios de Banach. Así como también damos ejemplos de operadores fuera de la clase de la familia de los operadores de convolución que resultan hipercíclicos. Estos ejemplos se presentan tanto en espacios de funciones holomorfas de finitas variables complejas y también en espacios de funciones holomorfas definidas en espacios de Banach de dimensión infinita.

Por otro lado, estudiamos pseudo órbitas de opeadores lineales. Decimos que $\{x_n\}_{n\in\mathbb{N}}$ es una $(\varepsilon_n)-$*pseudo órbita* para T si $d(x_{n+1}, T(x_n)) \leq \varepsilon_n$ para todo $n \in \mathbb{N}$. Esta definición cobra sentido cuando se permite cometer un error en cada paso de la iteración del sistema. Notemos que si $\varepsilon_n = 0$ para todo $n \in \mathbb{N}$, una (ε_n)-pseudo órbita es una órbita. Decimos que el operador T es (ε_n)-hipercíclico si existe una pseudo órbita densa para la sucesión de errores (ε_n). Estudiamos este concepto enmarcado dentro de la teoría de sistemas dinámicos lineales.

Palabras clave: Operadores Hipercíclicos, Pseudo Órbitas, Operadores de Convolución, Funciones Holomorfas.

Hypercyclicity in spaces of holomorphic functions and pseudo orbits of linear operators

Abstract

In this thesis we study several problems on the density of orbits of linear operators. A linear operator is said hypercyclic if admits a dense orbit. We can say that the spotlight is the behavior of successive iterations of a linear operator. In other words, dynamical systems associated to linear operators are studied. In the finite dimensional context the problem is relatively simple and it can be solved through the canonical Jordan form of a matrix. Hence, chaos is usually associated to non linear systems. However, in infinite dimensional spaces linear operators can be chaotic since new phenomena appears, such as the existence of dense orbits.

The first examples of hypercyclic operators came out in the context of analytic functions. Birkhoff in [Bir29] proved that for every $a \in \mathbb{C}$, $a \neq 0$, the translation operator on the space of one complex variable functions $(H(\mathbb{C}), \tau)$ with the compact-open topology, $\tau_a : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $\tau_a f(z) = f(z+a)$ is hypercyclic. MacLane in [Mac52], proved that the same occurs with the derivative operator on the space $H(\mathbb{C})$. Both results were generalized by a remarkable theorem due to Godefroy and Shapiro [GS91], who proved that every linear operator that commutes with the translation on $H(\mathbb{C}^N)$ and is not a scalar multiple of the identity is hypercyclic. This family of operators is known as the class of (non trivial) "convolution operators". In this thesis we study convolution operators defined on spaces of holomorphic functions on Banach spaces. In addition, we show examples of non convolution hypercyclic operators defined on spaces of holomorphic functions of finite complex variables and also for spaces of holomorphic functions defined on infinite dimensional Banach spaces.

On other side, we deal with pseudo orbits of linear operators. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a $(\varepsilon_n)-$*pseudo orbit* of T if $d(x_{n+1}, T(x_n)) \leq \varepsilon_n$ for any $n \in \mathbb{N}$. This definition becomes meaningful when a small measurement error is committed in each step of the iteration. We say that an operator T is (ε_n)-hypercyclic if admits a dense pseudo orbit for the error sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. We study this new concept framed on the theory of linear dynamical systems.

Keywords: Hypercyclic operators, Pseudo orbits, Convolution Operators, Holomorphic Functions.

Contents

Introducción

El estudio de la dinámica de operadores lineales definidos en espacios de Banach o de Fréchet es una rama moderna del análisis funcional que ha surgido a partir del trabajo de muchos autores. Probablemente, el inicio de su estudio de manera sistemática es la tesis doctoral de C. Kitai en 1982 [Kit82]. En particular, gran parte de la difusión de este tema de estudio debe ser atribuida a los trabajos de G. Godefroy y J. H. Shapiro [GS91] y K.-G. Grosse-Erdmann [GE99].

Para dar una idea sumamente simplificada del tema, podemos decir que el centro de atención es el comportamiento de las sucesivas iteraciones de un operador lineal. En otras palabras, se estudian sistemas dinámicos discretos asociados a operadores lineales. En el contexto finito dimensional este problema se puede resolver a través del estudio de la forma de Jordan asociada a una matriz, y los comportamientos son relativamente simples (de ahí que el caos se asocia naturalmente a sistemas no lineales). Sin embargo, en espacios de dimensión infinita los sistemas lineales pueden ser caóticos, ya que aparecen fenómenos nuevos, como por ejemplo la existencia de órbitas densas en todo el espacio. La palabra "hipercíclico" tiene su origen en la noción de "operador cíclico", ligado al *problema del subespacio invariante*. En este caso, los operadores hipercíclicos están ligados al problema de existencia de *subconjuntos invariantes*: ¿dado un operador lineal $T : X \to X$, es posible encontrar un subconjunto cerrado no trivial F tal que $T(F) \subset F$?

Sea X un espacio de Fréchet separable de dimensión infinita y $T : X \to X$ un operador lineal y continuo. Dado $x \in X$, la órbita de x por T es el conjunto definido por

$$Orb(x, T) = \{T^n x : n \geq 0\}.$$

El operador T se dice hipercíclico si existe $x \in X$ (vector hipercíclico) tal que $Orb(x, T)$ es densa en X.

Es importante notar que la existencia de operadores con esta propiedad es sólo posible en espacios de dimensión infinita ya que por ejemplo, si $T : X \to X$ es un operador lineal en un espacio de Fréchet y $T' : X' \to X'$ es su operador adjunto, la existencia de algún autovalor para T' garantiza que T no es hipercíclico. En los últimos años el estudio de la hiperciclicidad de operadores ha tenido un desarrollo importante, como referencia puede consultarse la bibliografía [BM09] y [GEPM11].

Los primeros ejemplos de operadores hipercíclicos surgieron en el contexto de la teoría de funciones analíticas. Así, en 1929, G. D. Birkhoff [Bir29] probó que para todo $a \in \mathbb{C}$ no nulo, el operador traslación en el espacio de funciones enteras de variable compleja $(H(\mathbb{C}), \tau)$ con la topología compacto-abierta, $T_a : H(\mathbb{C}) \to H(\mathbb{C})$ definido por $T_a f(z) = f(z + a)$ es hipercíclico, y en 1952, G. R. MacLane [Mac52], demostró que lo mismo ocurre con el operador de diferenciación en $H(\mathbb{C})$. Por supuesto, no existía aún la noción de hiperciclicidad, y el interés de estos trabajos no se centraba en la dinámica de los operadores sino en propiedades de las funciones analíticas. Estos resultados fueron generalizados por G. Godefroy y J. H. Shapiro en 1991 [GS91] quienes probaron que todo operador lineal y continuo $T : H(\mathbb{C}) \to H(\mathbb{C})$ que conmute con las traslaciones y no sea un múltiplo de la identidad es hipercíclico. Estos operadores se denominan operadores de convolución no triviales. Versiones similares de este resultado para espacios de Fréchet de funciones analíticas definidas en un espacio de Banach fueron obtenidas en [CDM07, Pet01, Pet06, AB99, BBFJ13].

El primer ejemplo de existencia de esta clase de operadores en espacios de Banach fue exhibido por S. Rolewicz en 1969 [Rol69]. En su trabajo prueba que para $1 \leq p < \infty$, si $S : \ell_p \to \ell_p$ es el operador shift a izquierda, $S(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$, entonces $T = \lambda S$ es hipercíclico para todo $\lambda \in \mathbb{C}$, $|\lambda| > 1$. También se plantea el problema de determinar si en todo espacio de Banach separable de dimensión infinita existen operadores hipercíclicos. Una respuesta positiva a esta pregunta fue dada independientemente por S. I. Ansari [Ans97] y L. Bernal [BG99] en 1997 y 1999 respectivamente. En el contexto de espacios de Fréchet, la existencia de operadores hipercíclicos fue probada por J. Bonet y A. Peris [BP98].

Un operador es hipercíclico en X si y sólo si es topológicamente transitivo sobre abiertos, es decir, para cada par de abiertos no vacíos U, $V \subset X$ existe $n \in \mathbb{N}$ tal que $T^n(U) \cap V \neq \emptyset$. En 2004, F. Bayart y S. Grivaux [BG04] generalizan esta definición imponiendo condiciones a la frecuencia con que la órbita visita el abierto e introducen el concepto de operador *frecuentemente hipercíclico*. El operador $T : X \to X$ es frecuentemente hipercíclico si existe $x \in X$ tal que para todo abierto $U \subset X$ resulta

$$\liminf_{N \to \infty} \frac{card\,\{n \leq N : T^n x \in U\}}{N} > 0.$$

Un resultado similar al obtenido por G. Godefroy y J. H. Shapiro fue probado por Bonilla y Grosse-Erdmann [BGE06] donde muestran que todo operador $T : H(\mathbb{C}^N) \to H(\mathbb{C}^N)$ de convolución no trivial es frecuentemente hipercíclico.

El concepto de hiperciclicidad frecuente está fuertemente relacionado al de ergodicidad dentro de la teoría de sistemas dinámicos medibles. Dada $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ una transformación que preserva medida en un espacio de probabilidad $(X, \mathcal{B}, \mu)$, decimos que T es ergódica respecto a μ si para todo par de conjuntos medibles $A, B \in \mathcal{B}$ de medida positiva, existe $n \geq 0$ tal que $T^n(A) \cap B \neq \emptyset$. De esta forma, si μ es estrictamente positiva sobre todos los abiertos de X y T es ergódico respecto a μ, se concluye que T es topológicamente transitivo y por lo tanto hipercíclico. También se definen otros conceptos más restrictivos que la ergodicidad. Decimos que T es strongly mixing respecto a μ si para todo par de conjuntos medibles $A, B \in \mathcal{B}$ de medida positiva, se satisface que

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Mediante una aplicación del teorema ergódico de Birkhoff se concluye que la existencia de una medida de probabilidad μ boreliana, T-invariante, estrictamente positiva sobre todos los abiertos de X tal que el sistema $(X, \mathrm{Borel}, \mu, T)$ resulte ergódico implica que T es frecuentemente hipercíclico y el conjunto de vectores frecuentemente hipercíclicos de T tiene μ-medida 1. Siguiendo esta linea de estudio se han obtenido diversos criterios [BM14, MAP13] que dan condiciones suficientes para asegurar que el operador T es strongly mixing respecto a una medida de probabilidad μ.

Podemos dividir el trabajo de investigación en dos partes. A continuación describimos los avances correspondientes.

Operadores (ε_n)-hipercíclicos

Como primera parte del trabajo definimos y estudiamos otro concepto relacionado a la ciclicidad de un operador lineal. Sea $T : X \to X$ un operador continuo definido en un espacio métrico (X, d). Sea $(\varepsilon_n)_{n\in\mathbb{N}}$ una sucesión acotada de números reales positivos. Decimos que $\{x_n\}_{n\in\mathbb{N}}$ es una (ε_n)-*pseudo órbita* para T si $d(x_{n+1}, T(x_n)) \leq \varepsilon_n$ para todo $n \in \mathbb{N}$. Esta definición cobra sentido cuando se permite cometer un error en cada paso de la iteración del sistema. Notar que si $\varepsilon_n = 0$ para todo $n \in \mathbb{N}$, una (ε_n)-pseudo órbita es una órbita. Decimos que el operador T es (ε_n)-hipercíclico si existe una pseudo órbita densa para la sucesión de errores (ε_n). Es claro que un operador hipercíclico es (ε_n)-hipercíclico para cualquier sucesión de errores (ε_n). A continuación enumeramos algunos resultados obtenidos:

- Probamos que el shift a derecha definido en c_0 o ℓ_p con $1 \leq p < \infty$ es $(n^{-1/2})$-hipercíclico. Luego, existen operadores (ε_n)-hipercíclicos que no son hipercíclicos.

- Probamos que si $(\varepsilon_n) \in \ell_1$, entonces el espectro puntual del adjunto de un operador (ε_n)-hipercíclico es vacío. De esta forma, probamos que si $(\varepsilon_n) \in \ell_1$, entonces no hay operadores (ε_n)-hipercíclicos en $\mathbb{C}^n$. Cabe recordar que no existen operadores hipercíclicos en espacios de dimensión finita.

- Probamos que dentro de la clase de operadores shift con pesos definidos en c_0 o ℓ_p con $1 \leq p < \infty$, ser (ε_n)-hipercíclico con $(\varepsilon_n) \in \ell_1$ es equivalente a ser hipercíclico.

- Estudiamos distintos criterios que brindan condiciones suficientes para que un operador sea (ε_n)-hipercíclico, en los casos en los que $(\varepsilon_n) \notin \ell_p$ con $1 \leq p < \infty$.

Los resultados obtenidos forman parte del trabajo [MPSa].

Operadores hipercíclicos en espacios de funciones holomorfas

Estudiamos operadores definidos en espacios de funciones holomorfas de tipo acotado sobre un espacio de Banach E, asociadas a distintos tipos de holomorfía U. Notamos a este espacio $H_{bU}(E)$. Más especificamente, dado un espacio de Banach E y un ideal de Banach de operadores U, que además sea un tipo de holomorfía, definimos el espacio $H_{bU}(E)$ como el conjunto de funciones enteras cuyo U-radio de convergencia en cero es infinito. Es decir, el conjunto de funciones holomorfas $f \in H(E)$ tales que $f = \sum_k \frac{d^k f(0)}{k!}$ en donde $d^k f(0) \in U_k(E)$ es un polinomio k-homogéneo para todo k y $\lim_{k\to\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{U_k}^{1/k} = 0$.

Decimos que el operador T definido en $H_{bU}(E)$ es de convolución si conmuta con las traslaciones. Probamos que todo operador de convolución que no es un múltiplo escalar de la identidad es strongly mixing con respecto a una medida gaussiana. Como consecuencia, estos operadores resultan frecuentemente hipercíclicos. Además, determinamos la existencia de funciones frecuentemente hipercíclicas con crecimiento exponencial, así como también la existencia de subespacios cerrados cuyos elementos no nulos son funciones frecuentemente hipercíclicas. Estos resultados forman parte del trabajo [MPS14].

El siguiente paso en la investigación fue estudiar la ciclicidad de operadores fuera de la clase de los operadores de convolución. R. Aron y D. Markose en el artículo [AM04] trabajan con operadores de la forma

$$f(z) \in H(\mathbb{C}) \mapsto f'(\lambda z + b) \in H(\mathbb{C});\ \text{con } \lambda, b \in \mathbb{C}.$$

En el mismo se prueba que si $|\lambda| \geq 1$ el operador es hipercíclico y que no lo es en el caso en que $|\lambda| < 1$ y $b = 0$. Si $\lambda \neq 1$, entonces el operador no es de convolución. Consideramos operadores análogos en $H(\mathbb{C}^N)$ y en $H_{bU}(E)$.
En el caso de operadores definidos en $H(\mathbb{C}^N)$ consideramos para $\alpha \in \mathbb{N}_0^N$, $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ y $b_1, \dots, b_N \in \mathbb{C}$,

$$Tf(z_1, \dots, z_N) = D^\alpha f(\lambda_1 z_1 + b_1, \dots, \lambda_N z_N + b_N),$$

en donde D^α denota el operador de diferenciación $\frac{\partial}{\partial z_1^{\alpha_1}} \circ \cdots \circ \frac{\partial}{\partial z_N^{\alpha_N}}$. Análogamente al caso en $H(\mathbb{C})$, si $\lambda_i \neq 1$ para algún valor de i, este operador no pertenece a la clase de operadores de convolución.
Probamos que si $\prod_i |\lambda_i|^{\alpha_i} \geq 1$, entonces T resulta frecuentemente hipercíclico; si existe $j \in \{1, \dots, N\}$ tal que $\lambda_j = 1$ y $b_j \neq 0$, entonces T resulta hipercíclico; y que T no es hipercíclico en otro caso. Con estos resultados completamos y generalizamos el trabajo [AM04] a espacios de funciones holomorfas de varias variables y además completamos todo el rango de posibilidades en el caso de una variable. Los resultados obtenidos forman parte del trabajo [MPSc].
En el caso de operadores definidos en espacios de funciones holomorfas de tipo acotado sobre un espacio de Banach asociadas a distintos tipos de holomorfía, $H_{bU}(E)$, aparecen nuevas dificultades provenientes de la complejidad propia del espacio en donde se define el operador. Sea E un espacio de Banach con base incondicional $(e_n)_{n\in\mathbb{N}}$. Definimos de forma similar para un multi-índice finito $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, una sucesión acotada $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell_\infty$ y un vector $b = \sum_{n\in\mathbb{N}} b_n e_n \in E$,

$$Tf(z) = D^\alpha f(\lambda z + b).$$

Bajo hipótesis adecuadas se tiene el siguiente teorema, sobre la ciclicidad del operador $T \in H_{bU}(E)$:

- Si $|\lambda^\alpha| \geq 1$ entonces T es strongly mixing en sentido gaussiano.
- Si $\|\lambda\|_\infty = 1$ y existe $i \in \mathbb{N}$ tal que $b_i \neq 0$ y $\lambda_i = 1$, entonces T es mixing.
- Si $\|\lambda\|_\infty = 1$ y $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \notin E''$, entonces T es mixing.
- Si U es AB-cerrado y $\zeta \in E''$, entonces T no es hipercíclico.

Estos resultados forman parte del trabajo [MPSb].

Introduction

The study of the dynamics of linear operators on Banach or Fréchet spaces is a modern branch on functional analysis that has emerged from the work of many authors. Probably, the systematic study of linear dynamics begins with the Ph.D. thesis of Kitai [Kit82]. In particular, part of the diffusion of this subject of study can be attributed to [GS91] and the survey paper [GE99].

We can say that the spotlight is the behavior of successive iterations of a linear operator. In other words, dynamical systems associated to linear operators are studied. In the finite dimensional context the problem is relatively simple and it can be solved through the canonical Jordan form of a matrix. Hence, chaos is usually associated to non linear systems. However, in infinite dimensional spaces linear operators can be chaotic since new phenomenon appears, such as the existence of dense orbits. The word "hypercyclic" has its origin in the notion of "cyclic operator", linked to the invariant subspace problem. In this case, hypercyclic operators are linked to the problem of invariant subset. ¿Given a linear operator $T : X \to X$, is it possible to find a closed non trivial invariant subset?

Let X be an infinite dimensional separable Fréchet space and $T : X \to X$ a linear operator. Given $x \in X$, the orbit of x under T is the set defined by

$$Orb(x, T) = \{T^n x : n \geq 0\}.$$

The operator T is hypercyclic if there exists some $x \in X$ (hypercyclic vector) such that $Orb(x, T)$ is dense in X. It is important to notice that the existence of operators with this property is only possible on infinite dimensional spaces since, for example, if $T : X \to X$ is a linear operator on a Fréchet space and $T^* : X^* \to X^*$ is the adjoint of T, then the existence of some eigenvalue for T^* guarantees that T is not hypercyclic. In the past years, hypercyclicity has had an important development, as principal references we give the following recent books [BM09] and [GEPM11].

The first examples of hypercyclic operators came out in the context of analytic functions. Birkhoff in [Bir29] proved that for every $a \in \mathbb{C}$ not zero, $\tau_a : H(\mathbb{C}) \to H(\mathbb{C})$ the translation operator on the space of one complex variable functions $(H(\mathbb{C}), \tau)$ with the compact-open topology, defined by $\tau_a f(z) = f(z + a)$ is hypercyclic. MacLane in [Mac52] proved that the same occurs with the derivative operator on the space $H(\mathbb{C})$. Both results were generalized by a remarkable theorem due to Godefroy and Shapiro [GS91], who proved that every linear operator that commutes with the translation on $H(\mathbb{C}^N)$ and is not a scalar multiple of the identity is hypercyclic. This family of operators is known as the class of (non trivial) "convolution operators". Extensions of the Godefroy Shapiro theorem were proved for Fréchet spaces of holomorphic functions defined on Banach spaces [CDM07, Pet01, Pet06, AB99, BBFJ13].

The first example of hypercyclic operator on Banach spaces was given by Rolewicz in [Rol69]. In the article it is proved that for $1 \leq p < \infty$, if $B : \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$ is the unilateral backward shift $B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$, then $T = \lambda B$ is hypercyclic for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$. Moreover, the problem of determining if every Banach space supports a hypercyclic operator is raised. A positive answer to this question was given independently by Ansari and Bernal in [Ans97] and [BG99] respectively. In the context of Fréchet spaces, the existence of hypercyclic operators was proved by Bonet and Peris in [BP98].

An operator is hypercyclic if and only if is topologically transitive, i.e. for every pair of open sets U, $V \subset X$ exist $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. It is immediate to see that a dense orbit visits any open set infinitely many times. In 2004, Bayart and Grivaux [BG04] defined a new class of operators imposing conditions on the number of returning times of an orbit to each open set. They introduce the concept of "frequently hypercyclic operators". An operator is said to be frequently hypercyclic if there exists an orbit such that for any open set $U \subset X$ the set of returning times of the orbit to U has positive lower density in $\mathbb{N}$.

In [BGE06], Bonilla and Grosse-Erdmann extended the Godefroy and Shapiro theorem by showing that every non trivial convolution operator is frequently hypercyclic. In addition, they proved the existence of frequently hypercyclic functions of exponential growth.

The concept of frequently hypercyclicity is strongly related to the one of ergodicity. Given a probability space $(X, \mathcal{B}, \mu)$ and a measure preserving map $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, we say that T is ergodic with respect to μ if for every pair of measurable sets $A, B \in \mathcal{B}$ with positive measure, there exists $n \geq 0$ such that $T^n(A) \cap B \neq \emptyset$. If μ is strictly positive on any nonempty open set, from the ergodicity of T we can conclude its hypercyclicity. Moreover, by a simple application of Birkhoff's ergodic theorem, we can also conclude the frequently hypercyclicity of T. Other notions of measurable dynamical systems will be important for us. A measure preserving map is strongly mixing with respect to μ if for every pair of measurable sets $A, B \in \mathcal{B}$ with positive measure, the following limit holds:

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Several criteria that give sufficient conditions for a linear operator to assure that the map is strongly mixing with respect to a probability measure has been developed [BM14, MAP13].

We can split the thesis in two different parts. In the first part we deal with pseudo orbits of linear operators, i.e. orbits of the operator with small error measurement at each step. We relate this new concept to the one of hypercyclicity and give some examples. In the second part we work with hypercyclic operators defined on spaces of holomorphic functions. We study convolution and non-convolution operators on different Fréchet spaces. Below we describe the principal results of each part.

Chain Transitive operators

We study a concept from topological dynamics and relate it to the hypercyclicity of a linear operator. Suppose that (X, d) is a metric space, $T : X \to X$ is a map on X and $(\varepsilon_n)_{n\in\mathbb{N}}$ is bounded sequence of positive real numbers. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a (ε_n)-*pseudo orbit* of T if $d(x_{n+1}, T(x_n)) \leq \varepsilon_n$ for any $n \in \mathbb{N}$. This definition becomes meaningful when a small measurement error is committed in each step of the iteration. We say that an operator T is (ε_n)-hypercyclic if admits a dense pseudo orbit for the error sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. It is clear that every hypercyclic operator is (ε_n)-hypercyclic for any error sequence. We investigate this class of operators and give partial answers to the question whether the converse of the previous observation is true. We state some of the results we obtain:

- We give some examples of (ε_n)-hypercyclic operators in the finite dimensional setting.
- We prove that if $(\varepsilon_n) \in \ell_1$, then the point spectrum of the adjoint if an (ε_n)-hypercyclic operator is empty. Therefore, if $(\varepsilon_n) \in \ell_1$, there are not (ε_n)-hypercyclic operators on $\mathbb{C}^n$.
- We prove that the unilateral backward shift B defined on $c_0(\mathbb{N})$ o $\ell_p(\mathbb{N})$ is (ε_n)-hypercyclic if and only if $(\varepsilon_n) \notin \ell_1$. Recall that B is not hypercyclic. We prove that for the class of unilateral weighted shift operators defined on $c_0(\mathbb{N})$ o $\ell_p(\mathbb{N})$, (ε_n)-hypercyclic operators with sumable error sequence are hypercyclic. We give an example of a bilateral weighted shift operator which is (n^{-2})-hypercyclic but not hypercyclic.
- We study several criteria that ensure that an operator is (ε_n)-hypercyclic for different classes of error sequences.

This results appear in [MPSa].

Hypercyclic operators on spaces of holomorphic functions

This part of the thesis is devoted to the study of operators on spaces of holomorphic functions. We consider holomorphic functions defined on several spaces as $\mathbb{C}$, $\mathbb{C}^N$ or even in a Banach space. A holomorphic function on a Banach space E is, locally, an infinite sum of homogeneous polynomials on E, its Taylor series expansion. In order to define the spaces of holomorphic functions on Banach spaces we need to deal with holomorphy types. Given a Banach space E, an holomorphy type $\mathfrak{A} = \{\mathfrak{A}_k(E)\}_{k\geq 0}$ is a sequence of Banach spaces of homogeneous polynomials on E. Holomorphy types determine spaces of holomorphic functions whose derivatives belong to a certain class of polynomials $\mathfrak{A}$ (where $\mathfrak{A}$ could make reference, for example, to the compact, nuclear or continuous polynomials) and satisfy certain growth conditions relative to the underlying spaces of homogeneous polynomials $\mathfrak{A}_k$. To wit, we define the space of holomorphic functions of bounded type associated to the holomorphy type $\mathfrak{A}$, $\mathcal{H}_{b\mathfrak{A}}(E)$ as the set of holomorphic functions f, $f = \sum_k \frac{d^k f(0)}{k!}$ such that $d^k f(0) \in \mathfrak{A}_k(E)$ is a k-homogeneous polynomial and the series has infinite radius of convergence, i.e. $\lim_{k\to\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k}^{1/k} = 0$.

Like in the finite dimensional case, we say that an operator on $\mathcal{H}_{b\mathfrak{A}}(E)$ is a convolution operator if commutes with every translation. In chapter 4 we prove that every non-trivial convolution operator is strongly mixing with respect a gaussian measure. Immediately, convolution operators are frequently hypercyclic. Besides we prove the existence of frequently hypercyclic functions of exponential growth and prove the existence of closed subspaces in which every non-zero vector is frequently hypercyclic. This results are in [MPS14].

Next, we focus on operators outside the class of convolution operators. Since, Birkhoff and MacLane examples are convolution operators, the question of whether there are hypercyclic operators which are not convolution operators is natural. Aron and Markose [AM04] give a positive answer to this question. Specifically, they consider the family of operators defined as

$$f(z) \in H(\mathbb{C}) \mapsto f'(\lambda z + b) \in H(\mathbb{C}); \text{ with } \lambda, b \in \mathbb{C}.$$

They prove that if $|\lambda| \geq 1$ then the operator is hypercyclic, and that the operator is not hypercyclic if $|\lambda| < 1$ and $b = 0$. Note that if $\lambda \neq 1$, then the operator lives outside the class of convolution operators.

In chapters 3 and 5 we work with analogous operators to the one studied by Aron and Markose defined on spaces of holomorphic functions such us $H(\mathbb{C}^N)$ and $\mathcal{H}_{b\mathfrak{A}}(E)$ respectively. In the case of holomorphic functions on $\mathbb{C}^N$ we fix $\alpha \in \mathbb{N}_0^N$, $\lambda_1, \dots, \lambda_N \in \mathbb{C}$ and $b_1, \dots, b_N \in \mathbb{C}$, and first we consider the operators

$$Tf(z_1, \dots, z_N) = D^\alpha f(\lambda_1 z_1 + b_1, \dots, \lambda_N z_N + b_N),$$

which are a composition of affine diagonal composition operators with the differentiation operator $D^\alpha = \frac{\partial}{\partial z_1^{\alpha_1}} \circ \cdots \circ \frac{\partial}{\partial z_N^{\alpha_N}}$. Like in the case of $H(\mathbb{C})$, if $\lambda_i \neq 1$ for some i, $1 \leq i \leq N$, the operator lives outside the class of convolution operators.

We prove that if $\prod_i |\lambda_i|^{\alpha_i} \geq 1$, then the operator T is strongly mixing with respect to a gaussian measure; if there exist $j \in \{1, \dots, N\}$ such that $\lambda_j = 1$ and $b_j \neq 0$, then the operator T is hypercyclic; otherwise, T is not hypercyclic. This result completes the study of the one dimensional case started in [AM04] and also characterizes completely the N-dimensional case. Also, we deal with operators in which the symbol of the composition part is not diagonal. We prove some partial results on the hypercyclicity of this operators in these cases. The results we obtain are part of the work [MPSc].

When dealing with holomorphic functions of bounded type on Banach spaces associated to a holomorphy type $\mathcal{H}_{b\mathfrak{A}}(E)$, new difficulties coming from the complexity of the space appear. Suppose that E is a Banach space with unconditional basis $(e_n)_{n\in\mathbb{N}}$. In a similar way, for a finite multi index $\alpha = (\alpha_n)_{n\in\mathbb{N}}$, a bounded sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell_\infty$ and a vector $b = \sum_{n\in\mathbb{N}} b_n e_n \in E$, it is possible to define

$$Tf(z) = D^\alpha f(\lambda z + b).$$

Under suitable hypothesis on the holomorphy type $\mathfrak{A}$, we have the following result on the cyclicity of the operator $T \in \mathcal{H}_{b\mathfrak{A}}(E)$:

- If $|\lambda^\alpha| \geq 1$ then T is strongly mixing with respect to a gaussian measure.
- If $\|\lambda\|_\infty = 1$ and there exist $i \in \mathbb{N}$ such that $b_i \neq 0$ and $\lambda_i = 1$, then T is mixing.
- If $\|\lambda\|_\infty = 1$ and $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \notin E''$, then T is mixing.
- If $\zeta \in E''$, then T is not hypercyclic.

Also, in the case of holomorphic functions of compact bounded type, we can avoid the additional hypothesis on the norm of the sequence λ. This results are included in [MPSb].

Chapter 1

Linear Dynamics

The aim of this first chapter is twofold: to give a reasonably short, yet significant and hopefully appetizing, sample of the type of questions with which we will be concerned and also to introduce some definitions and prove some basic facts that will be used throughout the whole thesis. For more on linear dynamics see [BM09, GEPM11].

1.1 Hypercyclic operators

The general frame will be a topological vector space (X, τ), over $\mathbb{R}$ or $\mathbb{C}$. The object of study will be continuous linear operators on (X, τ). We denote $\mathcal{L}(X)$ the space of continuous linear operators on X.

Definition 1.1.1. We say that a metric space (X, d) is a *F-space*, if (X, d) is a complete vector space over $\mathbb{R}$ or $\mathbb{C}$. In addition, if X is locally convex, we say that X is a *Fréchet space*.

An attractive feature of F-spaces is that one can make use of the Baire category theorem. This will be very important for us.

Theorem 1.1.2. [Baire's category Theorem] Let X be a complete metric space, then every countable intersection of dense open sets is dense.

Definition 1.1.3. We say that (X, T) is a *linear dynamical system*, if X is an F-space and $T \in \mathcal{L}(X)$.

We will study orbits defined by an operator when iterated on the space X.

Definition 1.1.4. Let (X, T) be a linear dynamical system. For $x \in X$, we define the *orbit of x under T* as the set

$$Orb(x, T) = \{T^n(x) : n \in \mathbb{N}_0\}.$$

Particularly, we are interested in determining the existence of dense orbits.

Definition 1.1.5. Let (X, T) be a linear dynamical system. We say that T is *hypercyclic* if there exists some $x \in X$ such that $Orb(x, T)$ is dense in X. In that case, we say that x is a *hypercyclic vector* of T. We denote $HC(T)$ the set of all hypercyclic vectors of T.

In a similar way, an operator is cyclic if there exists some $x \in X$ such that $\operatorname{span} Orb(x,T) = \mathbb{K}[T]x = \{P(T)x : P \in \mathbb{K}[t]\}$ is dense in X.

Remark 1.1.6. It is clear, that X must be separable in order to support a hypercyclic operator. Moreover, our first theorem which is due to Rolewicz, claims that hypercyclicity is a purely infinite-dimensional phenomenon [Rol69].

Theorem 1.1.7. There are no hypercyclic operators on finite dimensional spaces.

Our first characterization of hypercyclicity is a direct application of the Baire category theorem. This result was proved by Birkhoff in [Bir22], and it is often referred to as Birkhoff's transitivity theorem.

Definition 1.1.8. Let (X,T) be a linear dynamical system. We say that T is *topologically transitive* if for every pair of non-empty open sets U and V there exists $n \in \mathbb{N}$ such that $T^nU \cap V \neq \emptyset$.

Theorem 1.1.9 (Birkhoff's transitivity theorem)**.** Let (X,T) be a linear dynamical system and suppose that X is separable. Then, T is hypercyclic if and only if T is topologically transitive. In that case, $HC(T)$ is a dense G_δ subset.

When the operator T is invertible, it is clear that T is topologically transitive if and only if T^{-1} is. Thus, we can state

Corollary 1.1.10. Let (X,T) be a linear dynamical system. Suppose that X is separable and that T is invertible. Then T is hypercyclic if and only if T^{-1} is hypercyclic.

It is worth noting that T and T^{-1} do not necessarily share the same hypercyclic vectors. We continue with the first historic example, which is also due to Birkhoff in 1929 [Bir29]. Of course, this result was not framed in the theory of hypercyclic operators, but on the theory of holomorphic function.

Example 1.1.11 (Birkhoff 1929). Let $H(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$ endowed with the topology of uniform convergence on compact sets. For any non-zero complex number a, let $\tau_a : H(\mathbb{C}) \to H(\mathbb{C})$ be the translation operator defined by $\tau_a(f)(z) = f(z+a)$. Then τ_a is hypercyclic on $H(\mathbb{C})$.

For the proof we will need a useful approximation theorem, that we will use at several times in this thesis.

Theorem 1.1.12. [Runge's approximation theorem] Let K be a compact subset of $\mathbb{C}$ with connected complement. Then, any function f that is holomorphic in a neighbourhood of K can be uniformly approximated on K by polynomial functions.

Now we can prove that translations τ_a are hypercyclic on $H(\mathbb{C})$.

Proof. Since the space $H(\mathbb{C})$ is a separable Fréchet space, it is enough to show that τ_a is topologically transitive. Let U and V be two non-empty open subset of $H(\mathbb{C})$. There exist $\varepsilon > 0$, two complex closed disks L and K and two functions f, g in $H(\mathbb{C})$ such that

$$U \supset \{h \in H(\mathbb{C}) : \sup_{z \in K} |h(z) - f(z)| < \varepsilon\},$$

$$V \supset \{h \in H(\mathbb{C}) : \sup_{z\in L} |h(z) - g(z)| < \varepsilon\}.$$

We have that $\tau_a^n U \cap V \neq \emptyset$ if and only if there exist $h \in U$ such that $\tau_a^n h \in V$. Observe that

$$\tau_a^n h \in V \Leftrightarrow \sup_{z\in L} |\tau_a^n h(z) - g(z)| < \varepsilon,$$

$$\sup_{z\in L} |\tau_a^n h(z) - g(z)| = \sup_{z\in L+an} |h(z) - g(z - an)|.$$

So, let n be any natural number such that $K \cap (L + an) = \emptyset$. Since $K \cup (L + an)$ is compact in $\mathbb{C}$, with connected complement, from Runge's approximation theorem one can find $h \in H(\mathbb{C})$ such that

$$\sup_{z\in K} |h(z) - f(z)| < \varepsilon \quad \text{and} \quad \sup_{z\in L+an} |h(z) - g(z - an)| < \varepsilon.$$

Thus $h \in U$ and $\tau_a^n h \in V$, which shows that τ_a is topologically transitive. □

A useful tool for hypercyclicity is the following criterion. It gives sufficient conditions for a linear operator to be hypercyclic. It was first discovered by Kitai, and several variations of this criterion were proved. The following version is due to Bes [Bès98].

Theorem 1.1.13. [Hypercyclicity Criterion] Let (X, T) be a linear dynamical system where X is a separable F-space. Suppose that there exist an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$, two dense subsets D_1 and D_2 of X and a sequence of maps $S_{n_k} : D_2 \to X$, such that

1. $T^{n_k} x \to 0$, for any $x \in D_1$
2. $S^{n_k} y \to 0$, for any $y \in D_2$
3. $T^{n_k} S_{n_k} y \to y$, for any $y \in D_2$

Then, T is hypercyclic.

Remark 1.1.14. In fact, an operator that satisfies the Hypercyclicity Criterion is weakly mixing. It was proved by Bés and Peris that an operator T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is transitive. This condition is known as weakly mixing. The question of whether the conditions of the Hypercyclic Criterion are necessary for an operator to be hypercyclic was posed by Herrero [Her91]. Equivalently, is every hypercyclic operator necessarily weakly mixing? A negative answer was first given by De La Rosa and Read [dR09]. Then Bayart and Matheron were able to produce hypercyclic non-weakly-mixing operators on many classical spaces.

Remark 1.1.15. We point out that in the previous theorem the sets D_1 and D_2 need not to be subspaces. Also, the maps S_{n_k} are not assumed to be linear or continuous. In fact, we can replace the maps S_{n_k} and conditions 2) and 3) by the following conditions: for any $y \in D_2$ there is a sequence $(u_n)_{n\geq 0}$ in X with $u_0 = y$ such that $u_n \to 0$ and $T^n u_k = u_{n-k}$ if $n \leq k$.

With this powerful tool it can be shown the following examples of hypercyclic operators.

Example 1.1.16 (MacLane 1951). The derivative operator defined by $D(f) = f'$ is hypercyclic on $H(\mathbb{C})$.

Example 1.1.17 (Rolewicz 1969). Let $B : \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$, $1 \leq p < \infty$ be the unilateral backward shift operator defined by $B(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Then λB is hypercyclic for any scalar λ such that $|\lambda| > 1$.

From the Hypercyclicity Criterion the following result may be deduced. It is a criterion for hypercyclicity, due to Godefroy and Shapiro [GS91], based on the existence of sufficient amount of eigenvectors and eigenvalues.

Theorem 1.1.18. [Godefroy - Shapiro Criterion] Let (X, T) be a linear dynamical system where X is a separable F-space. Suppose that $\bigcup_{|\lambda|<1} Ker(T - \lambda)$ and $\bigcup_{|\lambda|>1} Ker(T - \lambda)$ both span dense subspaces of X. Then, T is hypercyclic.

To illustrate the Godefroy - Shapiro Criterion, we give the following example which generalizes Birkhoff and MacLane examples to a larger class of hypercyclic operators. Namely, the class of convolution operators. This class will be very important for us in the present thesis.

Definition 1.1.19. Let $T \in \mathcal{L}(H(\mathbb{C}^N))$. We say that T is a *convolution operator* if $T \circ \tau_a = \tau_a \circ T$ for any $a \in \mathbb{C}$. We say that T is a *trivial convolution operator* if T is a scalar multiple of the identity.

Observe that translation operators τ_a, with $a \neq 0$ and the derivative operator D are non trivial convolution operators.

In [GS91], the authors proved the following theorem on the hypercyclicity of non trivial convolution operators

Theorem 1.1.20. [Convolution Operators] Suppose that T is a non trivial convolution operator on $\mathcal{L}(H(\mathbb{C}^N))$, then T is hypercyclic.

Another important property of hypercyclicity is that it is preserved by linear conjugate. It is usually referred in the literature as the hypercyclic comparison principle.

Proposition 1.1.21. [Hypercyclic comparison principle] Let X and Y be separable F-spaces and $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$. Suppose that $SJ = JT$ for some continuous mapping $J : X \to Y$ of dense range. If S is hypercyclic then T is hypercyclic. Moreover, the map J sends hypercyclic vectors to hypercyclic vectors.

1.2 Spectral properties

In this section we show that hypercyclic operators have notorious spectral properties. We start with simple observations concerning the norm of a hypercyclic operator. We show that there are not hypercyclic operators in some classes of operators such as power bounded operators and compact operators. Also, we analyze the spectrum of a hypercyclic operator. We denote the spectrum and the point spectrum, i.e. the set of all eigenvalues of T, as $\sigma(T)$ and $\sigma_p(T)$, respectively.

It is clear that if T is contractive, i.e. $\|T\| \leq 1$, then every orbit is bounded. Thus, a hypercyclic operator cannot be contractive. Note also that a hypercyclic operator cannot be power bounded, i.e. $\sup \|T^n\| < \infty$, because again every orbit will be bounded. It is also not possible for a hypercyclic operator to be expansive, i.e. $\|Tx\| \geq \|x\|$ for every $x \in X$, because the orbit of any non-zero vector will stay far away from 0.

This remarks on the norm of the operator are related to properties on the spectrum of hypercyclic operators. We denote the (Banach) adjoint of T as T^*.

Proposition 1.2.1. Suppose that $T \in \mathcal{L}(X)$ is hypercyclic. Then, the point spectrum of the adjoint T^* is empty, i.e. $\sigma_p(T^*) = \emptyset$.

Remark 1.2.2. When X is a Banach space, the fact $\sigma_p(T^*) = \emptyset$ implies that $T - \alpha$ has dense range for all $\alpha \in \mathbb{K}$. Indeed,

$$R(T-\alpha)^{\perp} = Ker(T-\alpha)^* = Ker(T^* - \alpha) = \{0\}.$$

The last property is true for hypercyclic operators, even though the space X is not a Banach space. In fact if T is a hypercyclic operator, for every non-zero polynomial P, $P(T)$ has dense range. Thus, if $x \in HC(T)$, by Proposition 1.1.21, since $P(T)$ commutes with T and has dense range, it follows that $P(T)x \in HC(T)$. We have proved the following: if T is hypercyclic and x is a hypercyclic vector, then $Orb(x,T) \subset \mathbb{K}[T]x \setminus \{0\} \subset HC(T)$. We say that $\mathbb{K}[T]x$ is a *hypercyclic manifold* for T.

The following theorem, implies that a hypercyclic operator cannot be "partly" contractive nor expansive.

Theorem 1.2.3. Let X be a complex Banach space, and let $T \in \mathcal{L}(X)$ is hypercyclic. Then every connected component of the spectrum of T intersects the unit circle.

Finally, we state the following proposition on compact hypercyclic operators.

Proposition 1.2.4. Let X be a complex Banach space and $T \in \mathcal{L}(X)$ a compact operator. Then, T is not hypercyclic.

Proof. Assume that T is a compact hypercyclic operator. Thus, X must be infinite dimensional and then the spectrum of T is countable and contains $\{0\}$. But, $\{0\}$ is a connected component of $\sigma(T)$ that does not intersect the unit circle. $\square$

The same proposition holds for the real case using the complexification of the space and the previous proposition.

1.3 Other notions in linear dynamics

In this section we recall other concepts of the theory of linear dynamics that will be needed in the thesis. Also, we discuss some of the connections between ergodic theory and linear dynamics. First we give a brief summary of other forms of hypercyclicity.

Definition 1.3.1. A linear operator T on X is called *mixing* if for every pair of non void open sets U and V, there exists $N \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$ for every $n \geq N$.

A relatively new definition was given by Bayart and Grivaux in [BG06], an operator T is *frequently hypercyclic* if there exists a vector $x \in X$, called *frequently hypercyclic vector*, whose T-orbit visits each non-empty open set along a set of integers having positive lower density.

Definition 1.3.2. Let (X, T) be a linear dynamical system. We say that T is *frequently hypercyclic* if there exists a vector $x \in X$ such that for any non-empty open set $U \subset X$ the following holds

$$\liminf_{N \to \infty} \frac{card\,\{n \leq N : T^n x \in U\}}{N} > 0.$$

Now we show how ergodic theory can be related to linear dynamics. First we give some basic definitions.

Definition 1.3.3. Given a probability space $(X, \mathcal{B}, \mu)$ and a map $T : X \to X$ we say that T is a measure preserving transformation if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$.

The measure theoretic analogue to the notion of transitivity is ergodicity.

Definition 1.3.4. Given a probability space $(X, \mathcal{B}, \mu)$ and a measure preserving map $T : X \to X$ we say that T is ergodic with respect to μ is for every pair of measurable sets $A, B \in \mathcal{B}$ with positive measure, there exists $n \geq 0$ such that $T^n(A) \cap B \neq \emptyset$.

Remark 1.3.5. Equivalently, T is ergodic with respect to μ if for any $A \in \mathcal{B}$ such that $T(A) \subset A$, then $\mu(A) = 0\,or\,1$. Thus, the only measurable invariant sets are of null measure or of full measure.

If μ is strictly positive on any non-empty open set, i.e. μ is of full support, from ergodicity we can directly conclude hypercyclicity. But, from a simple application of Birkhoff's ergodic theorem [BM09, Theorem 5.3], we can also conclude frequent hypercyclicity. Other notions of measurable dynamical systems will be important for us.

Definition 1.3.6. A measure preserving map T is strongly mixing with respect to μ if for every pair of measurable set $A, B \in \mathcal{B}$ with positive measure, the following limit holds:

$$\lim_{n \to \infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

Several criteria that give sufficient conditions for a linear operator to assure that the map is strongly mixing with respect to a probability measure has been developed [BM14, MAP13].

Definition 1.3.7. A Borel probability measure on X is gaussian if and only if it is the distribution of an almost surely convergent random series of the form $\xi = \sum_0^\infty g_n x_n$, where $(x_n) \subset X$ and (g_n) is a sequence of independent, standard complex gaussian variables.

Definition 1.3.8. We say that an operator $T \in \mathcal{L}(X)$ is strongly mixing in the gaussian sense if there exists some gaussian T-invariant probability measure μ on X with full support such that T is strongly mixing with respect to μ.

We will use the following result, which is a corollary of a theorem due to Bayart and Matheron (see [BM14]). Essentially this theorem says that a large supply of eigenvectors associated to unimodular eigenvalues that are well distributed along the unit circle implies that the operator is strongly mixing in the gaussian sense.

Theorem 1.3.9. [Bayart, Matheron] Let X be a complex separable Fréchet space, and let $T \in \mathcal{L}(X)$. Assume that for any set $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the linear span of $\bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(T - \lambda)$ is dense in X. Then T is strongly mixing in the gaussian sense.

The following result, proved by Murillo-Arcila and Peris in [MAP13, Theorem 1], shows that operators defined on Fréchet spaces which satisfy the Frequent Hypercyclicity Criterion are strongly mixing with respect to an invariant Borel measure with full support.

Theorem 1.3.10. [Murillo-Arcila, Peris] Let X be a separable Fréchet space and $T \in \mathcal{L}(X)$. Suppose that there exists a dense subspace $X_0 \subset X$ such that $\sum_{n \in \mathbb{N}} T^n x$ is unconditionally convergent for all $x \in X_0$. Suppose further that there exist a sequence of maps $S_k : X_0 \to X$, $k \in \mathbb{N}$ such that $T \circ S_1 = Id$, $T \circ S_k = S_{k-1}$ and $\sum_k S_k(x)$ is unconditionally convergent for all $x \in X_0$. Then there exist a Borel probability measure μ in X, such that the operator T is strongly mixing respect to μ.

The hypothesis of the Theorem 1.3.10 implies the corresponding ones of the Theorem 1.3.9. So both Theorems allow us to conclude the existence of an invariant gaussian probability measure for linear operators of full support which are strongly mixing. Finally, the next proposition states that the existence of such measures is preserved by linear conjugation, just as in Proposition 1.1.21.

Proposition 1.3.11. Let X and Y be separable Fréchet spaces and $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$. Suppose that $SJ = JT$ for some linear mapping $J : X \to Y$ of dense range then, if T has an invariant Borel measure then so does S. Moreover, if T has an invariant Borel measure that is gaussian, strongly mixing, ergodic or of full support, then so does S.

Chapter 2

Chain Transitive Operators

Hypercyclicity is a phenomenon that deals with orbits of linear operators. This definition can be weakened if for example we consider pseudo orbits instead of orbits. Roughly speaking a pseudo orbit is an orbit in which a small error is committed in each iteration. Pseudo orbits have been studied in the context of dynamical systems on compact manifolds. Much of attention is focused in the pseudo orbit tracing property (abb. POTP) or shadowing, which is defined as follows: every δ-pseudo orbit with sufficiently small $\delta > 0$ can be arbitrarily close uniformly approximated by a true orbit (see [Pal00, Pil99]). The famous Shadowing Lemma essentially says that hyperbolicity implies the POTP. Other notions similar to shadowing have also been studied. For example, in [ENP97] the authors study the concepts of *limit-shadowing* and *ℓ_p-shadowing* in which instead of considering "uniform" pseudo orbits, c_0-pseudo orbits or ℓ_p-pseudo orbit are defined. Here the errors committed form a sequence that tends to zero or that is p-summable.

For linear operators on a Banach space shadowing is also related to hyperbolicity, moreover, in some cases both are equivalent (see for example [Omb93]). On the other hand, in the context of linear dynamics hyperbolicity is disjoint of hypercyclicity. In fact, a linear map is known to be hyperbolic if and only if its spectrum is disjoint with the unit circle, contrary to hypercyclicity in which any connected component of the spectrum of a hypercyclic operator must intersect the unit circle. In this chapter we define and study a concept related to pseudo orbits and hypercyclicity, namely the concept of (ε_n)-hypercyclicity. We say that an operator is (ε_n)-hypercyclic if it admits a dense pseudo orbit associated to the error sequence (ε_n). It is clear that a hypercyclic operator is (ε_n)-hypercyclic for any error sequence, since any true orbit is also a pseudo orbit.

When the error sequence $(\varepsilon_n) \in \ell_1$, we show that the point spectrum of the adjoint of an (ε_n)-hypercyclic operator is empty. Therefore, if $(\varepsilon_n) \in \ell_1$, there are not (ε_n)-hypercyclic operators on $\mathbb{C}^n$. However, we give some examples of (ε_n)-hypercyclic operators in the finite dimensional setting. We prove that the unilateral backward shift B defined on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ is (ε_n)-hypercyclic if and only if $(\varepsilon_n) \notin \ell_1$. Recall that B is not hypercyclic. We also prove that for the class of unilateral weighted shift operators defined on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$, (ε_n)-hypercyclic operators with summable error sequence are necessarily hypercyclic. However, (ε_n)-hypercyclicity with error sequence in ℓ_1 is not equivalent to hypercyclicity, we give an example of a bilateral weighted shift operator which is $(1/n^2)$-hypercyclic but not hypercyclic.

Finally, we study several criteria that ensure that an operator is (ε_n)-hypercyclic for different classes of error sequences.

The results of this chapter are included in [MPSa].

2.1 (ε_n)-hypercyclic operators

Definition 2.1.1. Let $T : X \to X$ be a continuous map defined on a metric space (X, d). Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a (ε_n)-*pseudo orbit* of T if $d(x_{n+1}, T(x_n)) \leq \varepsilon_n$ for all $n \in \mathbb{N}$. We will also say that $\{x_n\}$ is a *pseudo orbit of T for the error sequence* (ε_n).

Note that every orbit of T is a pseudo orbit. The concept of pseudo-orbit makes sense if in each step of the iteration we allow to have a small measurement error. We will always assume that the error sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers. We are interested in this new definition framed inside the theory of linear dynamics, in particular the relationship between this concept and the one of hypercyclicity.

Definition 2.1.2. Let $T : X \to X$ be a continuous map defined on a topological space X. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers. We say that T is (ε_n)-*hypercyclic*, if there exist a dense (ε_n)-pseudo orbit. We will also say that T is hypercyclic for the error sequence (ε_n).

It is clear that every hypercyclic operator is (ε_n)-hypercyclic for every error sequence (ε_n).

Remark 2.1.3. Generally we will assume that X is a separable normed space and that T is a continuous linear map on X. Notice that we do not require any hypothesis on the dimension of the space, even though there are not hypercyclic operators on finite dimensional spaces.

We start with the first examples and remarks. Our first example may seem trivial but illustrates a phenomenon that we will see repeatedly in the exposition.

Example 2.1.4. Let X be a separable normed space and consider I the identity map on X. Then I is (ε_n)-hypercyclic if and only if $(\varepsilon_n) \notin \ell_1$.

Proof. First suppose that $(\varepsilon_n) \in \ell_1$. Then, every (ε_n)-pseudo orbit $\{x_n\}_{n\geq 0}$, will satisfy that

$$\|x_n\| \leq \sum_{i=0}^{n-1} \|x_{i+1} - x_i\| + \|x_0\| \leq \sum_{i=0}^{n-1} \varepsilon_i + \|x_0\| \leq \|(\varepsilon_i)\|_1 + \|x_0\|.$$

Thus, every (ε_n)-pseudo orbit starting at x_0 remains bounded and it can not be dense.

Reciprocally, suppose that $(\varepsilon_n) \notin \ell_1$ and take $\{y_m\}_{m\in\mathbb{N}}$ a dense sequence in X with $y_m \neq 0$ for every $m \in \mathbb{N}$. We will prove that there is (ε_n)-pseudo orbit of I that contains $\{y_m\}_{m\in\mathbb{N}}$. Fix $x_0 = 0$. Consider the segment from x_0 to y_1. If $\|y_1\| < \varepsilon_1$, then we can directly take $x_1 = y_1$. If not, since $(\varepsilon_n) \notin \ell_1$, there exist some $N_1 \in \mathbb{N}$ such that

$$\sum_{i=1}^{N_1} \varepsilon_i < \|y_1\| \leq \sum_{i=1}^{N_1+1} \varepsilon_i.$$

Now, define for $1 \leq j \leq N_1$, $x_j = (\sum_{i=1}^{j} \varepsilon_i)\frac{y_1}{\|y_1\|}$ and $x_{N_1+1} = y_1$. Note that

$$\|x_j - x_{j-1}\| = \left\| \frac{\varepsilon_j}{\|y_1\|} y_1 \right\| = \varepsilon_j,$$

and

$$\|y_1 - x_{N_1}\| = \left\| y_1 - \left(\sum_{i=1}^{N_1} \frac{\varepsilon_i}{\|y_1\|} \right) y_1 \right\| = \|y_1\| - \sum_{i=1}^{N_1} \varepsilon_i \leq \varepsilon_{N_1+1}.$$

Next, we consider the segment from y_1 to y_2. Let $N_2 > N_1 + 1$ such that

$$\sum_{i=N_1+1}^{N_2} \varepsilon_i < \|y_2 - y_1\| \leq \sum_{i=N_1+1}^{N_2+1} \varepsilon_i.$$

Define for $N_1 + 2 \leq j \leq N_2$, $x_j = y_1 + (\sum_{i=N_1+1}^{j} \frac{\varepsilon_i}{\|y_2 - y_1\|})(y_2 - y_1)$ and $x_{N_2+1} = y_2$. Note that

$$\|x_j - x_{j-1}\| = \left\| \frac{\varepsilon_j}{\|y_2 - y_1\|}(y_2 - y_1) \right\| = \varepsilon_j,$$

and

$$\|y_2 - x_{N_2}\| = \left\| (y_2 - y_1) - \left(\sum_{i=N_1+1}^{N_2} \frac{\varepsilon_i}{\|y_2 - y_1\|} \right)(y_2 - y_1) \right\| = \|y_2 - y_1\| - \sum_{i=N_1+1}^{N_2} \varepsilon_i \leq \varepsilon_{N_2+1}.$$

Following inductively we get a (ε_n)-pseudo orbit for I that contains the dense set $\{y_m\}_{m\in\mathbb{N}}$, as we wanted to prove. □

Recall that neither a contractive nor expansive operators can be hypercyclic. Observe that there are contractive and expansive (ε_n)-hypercyclic operators. As we have seen, the identity map is (ε_n)-hypercyclic if the error sequence (ε_n) is not summable. However, if we discard the extreme cases ($\|T\| = 1$ for contractive operators and below bounded by 1 for expansive ones), we can prove that there are not (ε_n)-hypercyclic operators for the classes of "strictly" contractive and "strictly" expansive operators.

Proposition 2.1.5. Let X be a separable normed space and (ε_n) be an error sequence. Suppose that $T \in \mathcal{L}(X)$ is a continuous linear operator on X. Suppose that one of the following situations holds,

1. if $\|T\| < 1$, or
2. if T is bounded below by a positive constant $\alpha > 1$,

then T is not (ε_n)-hypercyclic.

Proof. Denote $\varepsilon = \|(\varepsilon_n)\|_\infty$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a (ε_n)-pseudo orbit for T.

For (1), suppose that $\|x_n\| \leq \frac{\varepsilon}{1-\|T\|}$, then

$$\|x_{n+1}\| \leq \|x_{n+1} - Tx_n\| + \|Tx_n\| \leq \varepsilon_n + \|T\|\|x_n\| \leq \varepsilon + \|T\|\frac{\varepsilon}{1 - \|T\|} = \frac{\varepsilon}{1 - \|T\|}.$$

Thus, once any pseudo orbit enters the ball $B\left(0, \frac{\varepsilon}{1-\|T\|}\right)$ does not come out from it. This proves that no pseudo orbit can be dense.

For (2), if $\|x_n\| \geq \frac{\varepsilon}{\alpha-1}$, then $\|x_{n+1}\| \geq \frac{\varepsilon}{\alpha-1}$, because

$$\|x_{n+1}\| \geq \|Tx_n\| - \varepsilon \geq \alpha\|x_n\| - \varepsilon \geq \alpha\frac{\varepsilon}{\alpha-1} - \varepsilon = \frac{\varepsilon}{\alpha-1}.$$

Then, once the pseudo orbit leaves the ball $B(0, \frac{\varepsilon}{\alpha-1})$ it does not return, which again proves that no pseudo orbit can be dense. $\square$

In the next proposition we show that there are not (ε_n)-hypercyclic operators if $\|T\| = 1$ and $(\varepsilon_n) \in \ell_1$. Thus, for the class of contractive operators in the extreme case of $\|T\| = 1$ we can prove a similar result if the error sequence is summable.

Proposition 2.1.6. Let X be a separable normed space. Suppose that T is a continuous linear operator on X such that $\|T\| = 1$ and that $(\varepsilon_n) \in \ell_1$. Then T is not (ε_n)-hypercyclic.

Proof. Suppose that T is (ε_n)-hypercyclic and that $\{x_n\}$ is a (ε_n)-pseudo orbit. Inductively,

$$\|x_{n+1}\| \leq \|x_0\| + \sum_{k=1}^{n+1} \varepsilon_k.$$

Indeed,

$$\|x_{n+1}\| \leq \|Tx_n\| + \varepsilon_{n+1} \leq \|x_n\| + \varepsilon_{n+1} \leq \|x_0\| + \sum_{k=1}^{n+1} \varepsilon_k \leq \|x_0\| + \|(\varepsilon_n)\|_1.$$

Then every pseudo orbit is bounded and can not be dense. $\square$

Now we give a characterization of (ε_n)-hypercyclic operators which is analogue to topological transitivity for hypercyclic operators. First we need some definitions and then we prove our main result of this section, which will provide a criterion to prove that an operator is (ε_n)-hypercyclic.

Definition 2.1.7. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Suppose that $(\varepsilon_n) \in \ell_\infty$ is a bounded error sequence of positive real numbers. Given an open set $U \subset X$ and $k, m \in \mathbb{N}$ we define

$$T^{-m}_{(\varepsilon_n)_k} U := T^{-1}\left[\ldots\left[T^{-1}\left[T^{-1}\left[U + B(0, \varepsilon_{k+m-1})\right] + B(0, \varepsilon_{k+m-2})\right]\ldots\right] + B(0, \varepsilon_k)\right]$$

and

$$T^{m}_{(\varepsilon_n)_k} U := T\left[\ldots\left[T\left[T(U) + B(0, \varepsilon_k)\right] + B(0, \varepsilon_{k+1})\right]\ldots\right] + B(0, \varepsilon_{k+m})$$

This sets are appropriate to study finite pseudo orbits.

Proposition 2.1.8. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Then,

1. $x \in T^{-m}_{(\varepsilon_n)_k} U$ if and only if there exists a finite pseudo orbit $x_0 = x, x_1, \ldots, x_{m-1}, x_m$ such that $x_m \in U$ and $\|x_j - Tx_{j-1}\| < \varepsilon_{j+k-1}$ for every $j = 1, \ldots, m$.

2. $x \in T^{m}_{(\varepsilon_n)_k} U$ if and only if there exists a finite pseudo orbit $x_0, x_1, \ldots, x_{m-1}, x_m = x$ such that $x_0 \in U$ and $\|x_j - Tx_{j-1}\| < \varepsilon_{j+k-1}$ for every $j = 1, \ldots, m$.

Proof. We will prove (1). The proof of (2) is analogous. If $m = 1$ and $x_0 \in T^{-1}_{(\varepsilon_n)_k} U = T^{-1}[U + B(0, \varepsilon_k)]$. We get that there exist $x_1 \in U$ such that $x_1 - Tx_0 \in B(0, \varepsilon_k)$. If $m \geq 2$ and $x_0 \in T^{-m}_{(\varepsilon_n)_k} U = T^{-1}[T^{-(m-1)}_{(\varepsilon_n)_{k+1}} U + B(0, \varepsilon_k)]$. We get that there exist $x_1 \in T^{-(m-1)}_{(\varepsilon_n)_{k+1}} U$ such that $x_1 - Tx_0 \in B(0, \varepsilon_k)$. Since $x_1 \in T^{-(m-1)}_{(\varepsilon_n)_{k+1}} U$ by inductive hypothesis we get that there exist x_1, $\dots$, $x_m \in U$ such that $\|x_j - Tx_{j-1}\| < \varepsilon_{j+k-1}$ for every $j = 1, \dots, m$. □

The proper approach to the concept of transitivity differs from the original one because of the errors that may be committed in each iteration. The correct definition reads as follows.

Definition 2.1.9. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Suppose that (ε_n) is an error sequence. We say that T is (ε_n)-chain transitive if for every pair of non void open sets U and V and every integer $k_0 \in \mathbb{N}$, there exist two integer $m \in \mathbb{N}$ and $k \geq k_0$ such that $T^{-m}_{(\varepsilon_n)_k} U \cap V \neq \emptyset$.

Theorem 2.1.10. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Suppose that (ε_n) is a error sequence. Consider the following statements:

1. for every pair of non void open sets U and V and every integer $k_0 \in \mathbb{N}$, exists $m \in \mathbb{N}$ such that $T^{-m}_{(\varepsilon_n)_{k_0}} U \cap V \neq \emptyset$.
2. there exist a countable dense set D such that for every $y \in D$, $k \in \mathbb{N}$, and $r_1,\ r_2 > 0$, there exist positive integers m and l such that $T^{-m}_{(\varepsilon_n)_k} B(0, r_1) \cap B(y, r_2) \neq \emptyset$ and $T^{-l}_{(\varepsilon_n)_k} B(y, r_2) \cap B(0, r_1) \neq \emptyset$,
3. T is (ε_n)-hypercyclic,
4. T is (ε_n)-chain transitive,
5. for every pair of non void open sets U and V and every integer $k_0 \in \mathbb{N}$, exist two positive integer $m \in \mathbb{N}$ and $k \geq k_0$ such that $T^{m}_{(\varepsilon_n)_k} U \cap V \neq \emptyset$.

Then the following implications hold: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5). In addition, if the error sequence is non increasing, then all statements are equivalent.

Proof. It is clear that (1) $\Rightarrow$ (2). For the implication (2) $\Rightarrow$ (1), take two non void open sets U and V and consider y_U,$y_V \in D$ and $r_U,\ r_V > 0$, such that $B(y_U, r_U) \subset U$, and $B(y_V, r_V) \subset V$. In order to obtain a pseudo orbit from V to U with errors bounded by the sequence $(\varepsilon_n)_{k_0}$ we just need to join two pseudo orbits one from V to a neighbour of 0 with a pseudo orbit from this neighbour that ends in U. In fact, by (2), there exists $l \in \mathbb{N}$ such that $T^{-l}_{(\varepsilon_n)_{k_0}} B(y_V, r_V) \cap B(0, 1) \neq \emptyset$. Suppose that the last element of this pseudo orbit is $x \in B(0, 1)$. Then, by (2) again, there exists $m \in \mathbb{N}$ such that $T^{-m}_{(\varepsilon_n)_k} B(x, \varepsilon_{m+k_0}) \cap B(y_U, r_U) \neq \emptyset$, where we have used the fact that D is a dense set and $B(x, \varepsilon_{m+k_0})$ is a non void open set.

To prove (2) $\Rightarrow$ (3), we will construct a dense pseudo orbit for T associated to the error sequence (ε_n), by sticking finite pseudo orbits that gets close enough to the point on the dense set D. Note that condition (1) implies that there exist finite pseudo orbits that goes from the ball $B(0, r_2)$ to the ball $B(y, r_2)$ with errors bounded by the sequence $(\varepsilon_n)_{n \geq k}$ for *every* $k \in \mathbb{N}$, and the same happens if we change $B(0, r_2)$ by $B(y, r_2)$.

Fixed an enumeration of D, $D = \{y_n\}_{n\in\mathbb{N}}$. First take a finite pseudo orbit that starts in $B(0,1)$ and ends in $B(y_1,1)$ with errors bounded by the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$. Suppose that this finite pseudo orbit has l_1 elements and denote it's last element x_{l_1}. Now, we consider open balls such that $B(y_{k_1},r_1) \subset B(x_{l_1},\varepsilon_{l_1})$ for some $y_{k_1} \in D$ and $r_1 > 0$. By hypothesis, we can take a finite pseudo orbit that starts in $B(y_{k_1},r_1)$ and ends in $B(0,1)$ with errors bounded by the sequence $(\varepsilon_n)_{n\geq l_1}$. Suppose that the finite pseudo orbit we have constructed so far has l_2 elements and denote it's last element x_{l_2}. Again we consider open balls such that $B(y_{k_2},r_2) \subset B(x_{l_2},\varepsilon_{l_2})$ for some $y_{k_2} \in D$ and $r_2 > 0$. Now, we can take a finite pseudo orbit that starts in $B(y_{k_2},r_2)$ and ends in $B(y_2,1/2)$ with errors bounded by the sequence $(\varepsilon_n)_{n\geq l_2}$. Following in this way, we prove the existence of an (ε_n)-pseudo orbit for T, that we denote $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_n\}_{n\in\mathbb{N}} \cap B(y_k,1/k) \neq \emptyset$ for every $k \in \mathbb{N}$.

It is clear that the pseudo orbit $\{x_n\}_{n\in\mathbb{N}}$ is dense in X. In fact, let U be an open subset of X. There exists $y \in D$ and $\delta > 0$ such that $B(y,\delta) \subset U$. Since D is dense, we have that $D \cap B(y,\delta)$ is infinite. Thus, there exist $m \in \mathbb{N}$, $m > \frac{2}{\delta}$ such that $y_m \in B(y,\frac{\delta}{2})$. This means that $B(y_m,1/m) \subset B(y,\delta)$: if $z \in B(y_m,1/m)$

$$\|z-y\| \leq \|z-y_m\| + \|y_m - y\| < \frac{1}{m} + \frac{\delta}{2} < \delta.$$

Then, since $\{x_n\}_{n\in\mathbb{N}} \cap B(y_m,1/m) \neq \emptyset$, we get that $\{x_n\}_{n\in\mathbb{N}} \cap U \neq \emptyset$.

For the implication (3) $\Rightarrow$ (4), we proceed directly. Suppose that $(x_n)_n$ is a dense pseudo orbit associated to the error sequence $(\varepsilon_n)_n$. Let U and V be two non void open sets and $k_0 \in \mathbb{N}$. By density, there exist $m_u > m_v > k_0$ such that $x_{m_u} \in U$ and $x_{m_v} \in V$. Thus, we get that there exists a finite pseudo orbit that starts at V, ends in U in $m_u - m_v$ iterations with errors bounded by $(\varepsilon_n)_{n\leq m_v}$. Therefore, $T^{-(m_u-m_v)}_{(\varepsilon_n)_{m_v}} U \cap V \neq \emptyset$.

The equivalence between (4) $\Leftrightarrow$ (5) is Proposition 2.1.8.

Finally, is the error sequence is non increasing is clear that (5) $\Rightarrow$ (1) because if $k \geq k_0$ then a pseudo orbit from V to U with errors bounded by the sequence $(\varepsilon_n)_{n\geq k}$ will also be bounded by the sequence $(\varepsilon_n)_{n\geq k_0}$. ☐

Definition 2.1.11. We say that an operator T is power bounded if $\sup_n \|T^n\| < \infty$.

Lemma 2.1.12. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Suppose that (ε_n) is a error sequence. Suppose also that T is power bounded, $(\varepsilon_n) \notin \ell_1$ and there exist a countable dense set D such that for every $k \in \mathbb{N}$, $y \in D$ and $r_1,\ r_2 > 0$, there exists an integers m such that $T^{-m}_{(\varepsilon_n)_k} B(y,r_2) \cap B(0,r_1) \neq \emptyset$, then T is (ε_n)-hypercyclic.

Proof. By condition (1) of Theorem 2.1.10, it is enough to prove that there exists $l \in \mathbb{N}$ such that $T^{-l}_{(\varepsilon_n)_k} B(0,r_1) \cap B(y,r_2) \neq \emptyset$, for every $k \in \mathbb{N}$, $y \in X$ and $r_1,\ r_2 > 0$. In order to prove that there exist a finite pseudo orbit that goes from $B(y,r_2)$ to $B(0,r_1)$, we will shrink the norm of the element of the pseudo orbit in each iteration by a factor given by the error. Fix $y \in X$ and $k \in \mathbb{N}$. Suppose that $M := \sup_n \|T^n\| < \infty$. Let us denote $x_k = y$ and $n \in \mathbb{N}$ such that $M\|y\| = \sum_{i=1}^{n} \varepsilon_{k+i-1}$. Consider the pseudo orbit of the form

$$x_{n+k-1} = T^n(y) + T^{n-1}(\varepsilon_k v_k) + \cdots + \varepsilon_{n+k-1} v_{n+k-1},$$

where the vectors $v_l = \frac{T^{l-k+1}y}{\|T^{l-k+1}y\|}$ are the directions of the errors committed at each iteration for $l = k, \ldots, n+k-1$. We have that

$$\|x_{n+k-1}\| = \|T^n y\| \left(1 - \sum_{i=1}^{n} \frac{\varepsilon_{k+i-1}}{\|T^i y\|}\right) \leq M\|y\| \left(1 - \frac{1}{M\|y\|} \sum_{i=1}^{n} \varepsilon_{k+i-1}\right) = 0.$$

As we wanted to see, $y = x_k, \ldots, x_{n+k-1} = 0$ is an (ε_j)-pseudo orbit with errors starting at ε_k. □

Remark 2.1.13. Since every operator T with $\|T\| \leq 1$ is power bounded, the previous Lemma holds if $\|T\| \leq 1$, $(\varepsilon_n) \notin \ell_1$ and there exist a countable dense set D such that for every $k \in \mathbb{N}$, $y \in D$ and $r_1,\ r_2 > 0$, there exists an integers m such that $T^{-m}_{(\varepsilon_n)_k} B(y, r_2) \cap B(0, r_1) \neq \emptyset$.

Also, we can give a similar lemma relating the spectral radius of the operator with the radius of convergence of the power series induced by the error sequence.

Lemma 2.1.14. Let X be a separable normed space and $T \in \mathcal{L}(X)$. Suppose that (ε_n) is a error sequence. Suppose also that $r(T) < \overline{\lim}_{i\to\infty}(\varepsilon_{k+i-1})^{1/i}$ for every $k \in \mathbb{N}$, and there exist a countable dense set D such that for every $k \in \mathbb{N}$, $y \in D$ and $r_1,\ r_2 > 0$, there exists an integers m such that $T^{-m}_{(\varepsilon_n)_k} B(y, r_2) \cap B(0, r_1) \neq \emptyset$, then T is (ε_n)-hypercyclic.

Proof. The proof follows the same lines of the last proof. By condition (1) of Theorem 2.1.10, it is enough to prove that there exists $l \in \mathbb{N}$ such that $T^{-l}_{(\varepsilon_n)_k} B(0, r_1) \cap B(y, r_2) \neq \emptyset$, for every $k \in \mathbb{N}$, $y \in X$ and $r_1,\ r_2 > 0$. In order to prove that there exist a finite pseudo orbit that goes from $B(y, r_2)$ to $B(0, r_1)$, we will shrink the norm of the element of the pseudo orbit in each iteration by a factor given by the error. Fix $y \in X$ and $k \in \mathbb{N}$. Since $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, there exists positive constants $C > 0$ and $\delta > 0$, such that $\|T^n\| \leq C(r(T) + \delta)^n$ for all $n \in \mathbb{N}$. Let us denote $x_k = y$. Since $r(T) < \overline{\lim}_{i\to\infty}(\varepsilon_{k+i-1})^{1/i}$ there exist $n \in \mathbb{N}$ such that

$$C\|y\| = \sum_{i=1}^{n} \varepsilon_{k+i-1} \left(\frac{1}{r(T) + \delta}\right)^i.$$

As before, consider the pseudo orbit of the form

$$x_{n+k-1} = T^n(y) + T^{n-1}(\varepsilon_k v_k) + \cdots + \varepsilon_{n+k-1} v_{n+k-1},$$

where the vectors $v_l = \frac{T^{l-k+1}y}{\|T^{l-k+1}y\|}$ are the directions of the errors committed at each iteration for $l = k, \ldots, n+k-1$. We have that

$$\|x_{n+k-1}\| = \|T^n y\| \left(1 - \sum_{i=1}^{n} \frac{\varepsilon_{k+i-1}}{\|T^i y\|}\right) \leq C(r(T)+\delta)^n \left(1 - \frac{1}{C\|y\|} \sum_{i=1}^{n} \varepsilon_{k+i-1} \left(\frac{1}{r(T) + \delta}\right)^i\right) = 0.$$

As we wanted to see, $y = x_k, \ldots, x_{n+k-1} = 0$ is an (ε_j)-pseudo orbit with errors starting at ε_k. □

2.2 Some non trivial examples

In this section we give our first non trivial examples of (ε_n)-hypercyclic operators. We start by analyzing linear maps acting in finite dimensional spaces over $\mathbb{R}$ or $\mathbb{C}$. Even though the ideas are relatively simple, the behavior of the operators is rather complicated and unexpected. We will start with diagonal matrices and Jordan blocks. The third example is the unilateral backward shift acting on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$, $1 \leq p < \infty$. In all three examples, we will see that the operators are (ε_n)-hypercyclic for some error sequence, but if we consider smaller error sequences, then no pseudo orbit can be dense.

Diagonal matrices Our first example deals with diagonal matrices in $\mathbb{C}^N$. Given $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, N$, consider the linear map given by the following matrix

$$T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

Proposition 2.2.1. Suppose that $(\varepsilon_n)_{n\in\mathbb{N}}$ is an error sequence. Given $\lambda_j \in \mathbb{C}$, $j = 1, \ldots, N$, consider the diagonal linear map T with diagonal entries λ_i acting on $\mathbb{C}^N$. Then, T is (ε_n)-hypercyclic if and only if $(\varepsilon_n) \notin \ell_1$ and $|\lambda_j| = 1$ for every $j = 1, \ldots, N$.

Proof. Suppose first that T is (ε_n)-hypercyclic. We will see later in Proposition 2.3.1 that the eigenvalues of the adjoint of T have modulus 1. Thus $|\lambda_j| = 1$ for every $j = 1, \ldots, N$. Observe that we may suppose that $\mathbb{C}^N$ is endowed with the euclidean norm. Since $\|T\| = 1$, by Proposition 2.1.6, we have that $(\varepsilon_n) \notin \ell_1$.

Now suppose that $(\varepsilon_n) \notin \ell_1$ and $|\lambda_j| = 1$ for every $j = 1, \ldots, N$. Again we may suppose that $\mathbb{C}^N$ is endowed with the euclidean norm. Since $\|Tx\| = \|x\|$ for every $x \in \mathbb{C}^N$ and $(\varepsilon_n) \notin \ell_1$, by Remark 2.1.13, it is enough to show that for every $k \in \mathbb{N}$, $y \in \mathbb{C}^N$ and $r_1,\ r_2 > 0$, there exists an integers m such that $T^{-m}_{(\varepsilon_n)_k} B(y, r_2) \cap B(0, r_1) \neq \emptyset$. That is to say, we need to prove that there exist a finite pseudo orbit that starts at 0, ends in $B(y, r_2)$ with errors bounded by the sequence $(\varepsilon_n)_{n\geq k}$. We will prove that there exist a finite pseudo orbit that ends exactly at y. In fact, fix $k \in \mathbb{N}$, $y \in \mathbb{C}^N$. Since, $(\varepsilon_n) \notin \ell_1$, there exists $m \in \mathbb{N}$ such that

$$\sum_{j=k}^{m+k-1} \varepsilon_j < \|y\| \leq \sum_{j=k}^{m+k} \varepsilon_j.$$

We can suppose that $\|y\| = \sum_{j=k}^{m+k} \varepsilon_j$. If $\|y\| < \sum_{j=k}^{m+k} \varepsilon_j$, then we can change ε_{m+k} by

$$\hat{\varepsilon}_{m+k} := \|y\| - \sum_{j=k}^{m+k-1} \varepsilon_j < \varepsilon_{m+k}.$$

We will take at each step j, $k \leq j \leq k+m$, an error $\varepsilon_j v_j$ with $\|v_j\| = 1$. Then, the finite pseudo orbit starting at 0 at step k arrives to the following vector at step $k+m$:

$$\varepsilon_k T^m(v_k) + \varepsilon_{k+1} T^{m-1}(v_{k+1}) + \ldots \varepsilon_{m+k-1} T(v_{m+k-1}) + \varepsilon_{m+k} v_{m+k}.$$

Consider for $j = k, \dots, m+k$, the following unitary vector that is the direction of the error committed at each step

$$v_j := \frac{1}{\|y\|}\left(\overline{sg(\lambda_1^{m+k-j})}y_1, \dots, \overline{sg(\lambda_N^{m+k-j})}y_N\right).$$

Then this pseudo orbit after m iterations ends exactly at y:

$$\begin{aligned}\sum_{j=k}^{m+k} \varepsilon_j T^{m+k-j}(v_j) &= \sum_{j=k}^{m+k} \varepsilon_j \frac{1}{\|y\|}\left(\lambda_1^{m+k-j}\overline{sg(\lambda_1^{m+k-j})}y_1, \dots, \lambda_N^{m+k-j}\overline{sg(\lambda_N^{m+k-j})}y_N\right) \\ &= \sum_{j=k}^{m+k} \varepsilon_j \frac{1}{\|y\|} y = y\end{aligned}$$

$\square$

A non diagonal matrix in $\mathbb{K}^2$ Consider the linear map T in $\mathbb{K}^2$, where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$, defined by

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

We will study the (ε_n)-hypercyclicity of T for several error sequences. We are particularly interested on error sequences $\varepsilon_n = 1/n^p$, with $0 \leq p \leq \infty$.

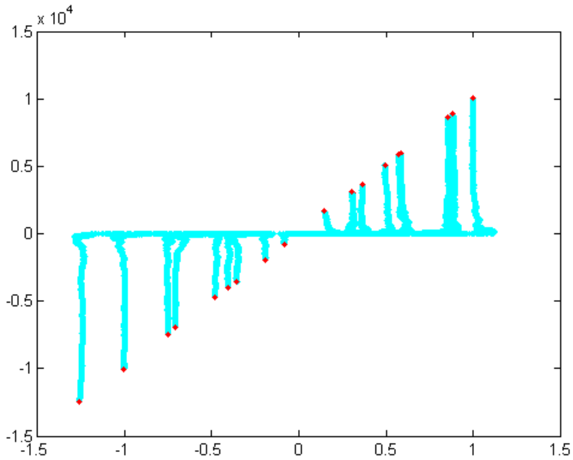

Figure 2.1: 20 finite random pseudo orbits with errors $1/n$

In Figure 2.1, we observe 20 random pseudo orbit of T starting at $(0,0)$ with error sequence $1/n$. It can be seen, that the red points (ending points after 10^4 iterations) are close to the line with equation $y = 10^4 x$. This motivates the next proposition.

Proposition 2.2.2. The map T is not $(1/n)$-hypercyclic.

Proof. Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is a pseudo orbit for T with errors $1/n$. Then there exist $\alpha_n \in \mathbb{R}$, $\alpha_n \leq 1/n$ and unitary vectors v_n, such that

$$x_{n+1} = T^{n+1}x_0 + \sum_{k=0}^{n} \alpha_{k+1}T^{n-k}(v_{k+1}).$$

Since the powers of T are

$$T^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

we get that

$$x_{n+1}(1) = x_0(1) + \sum_{k=0}^{n} \alpha_{k+1}v_{k+1}(1),$$

$$x_{n+1}(2) = (n+1)x_0(1) + x_0(2) + \sum_{k=0}^{n} \alpha_{k+1}((n-k)v_{k+1}(1) + v_{k+1}(2)).$$

Thus,

$$\begin{aligned} |x_{n+1}(2) - nx_{n+1}(1)| &= |x_0(1) + x_0(2) + \sum_{k=0}^{n} \alpha_{k+1}(-kv_{k+1}(1) + v_{k+1}(2))| \\ &\leq |x_0(1) + x_0(2)| + \sum_{k=0}^{n} \frac{1}{k+1}\sqrt{k^2+1} \leq n + C(x_0). \end{aligned}$$

Where we considered, without lost of generality, the euclidean norm on $\mathbb{K}^2$. Let x be any complex number, $\varepsilon > 0$. Suppose that $x_n \in B((x, -x), \varepsilon)$, then

$$\begin{aligned} (n+1)|x| &= |-x-nx| \leq |-x-x_n(2)| + |x_n(2) - nx_n(1)| + |nx_n(1) - nx| \\ &\leq (n+1)\varepsilon + n + C(x_0). \end{aligned}$$

Therefore, $|x| \leq \varepsilon + 1 + C(x_0)$. Which proves that no pseudo orbit with error sequence $(1/n)$ can be dense. □

Figure 2.2 represents the end points of 20 finite random pseudo orbits of T, considering a constant error sequence. As we can appreciate, the pseudo orbits seem to be more dispersive. This is confirmed by the following proposition.

Proposition 2.2.3. If the error sequence is constant, i.e. $\varepsilon_n = \varepsilon$ for all $n \in \mathbb{N}$, then T is (ε_n)-hypercyclic.

Proof. By Theorem 2.1.10 and Proposition 2.1.8 it suffices to show that for any $k \in \mathbb{N}$ there is a pseudo orbit starting at time k, from $(0,0)$ to (x,y) and from (x,y) to $(0,0)$, where (x,y) is an arbitrary vector. Note that since the error sequence is constant, it is sufficient to consider $k = 1$. Note that every point of $\mathbb{C}^2$ of the form $(0,y)$ is a fixed point of T. Having this in mind, in order to construct a pseudo orbit from $(x,y) \in \mathbb{C}^2$ to $(0,0)$, we can first use the errors to shrink the modulus of the first coordinate of the vectors with the goal of reaching the vertical

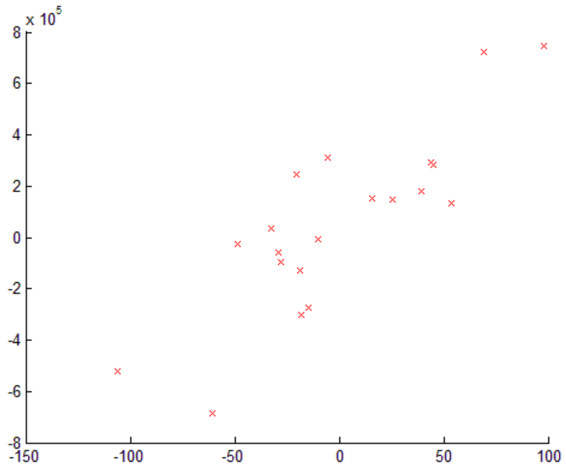

Figure 2.2: end points of 20 finite pseudo orbits with constant error sequence

axis. Once we arrive to the vertical axis, since every point in the vertical axis is a fixed point, we can shrink the modulus of the second coordinate until reaching $(0,0)$.

To make the reverse path, we need to construct a finite pseudo orbit starting at $(0,0)$ that ends in the point (x,y). This pseudo orbit will first leave the origin through the vertical axis to a vector of the form $(0,z)$ with errors that move the pseudo orbit in the second coordinate, and then will move from $(0,z)$ to (x,y) with errors that move the pseudo orbit in the first coordinate. More precisely, suppose $(x,y) = (|x|e^{i\theta}, |y|e^{i\varphi})$. Define the following variables:

$$M := \left\lceil \frac{|x|}{\varepsilon} \right\rceil, \qquad z := \frac{-1}{2}\varepsilon e^{i\theta} M(M-1), \qquad N := \left\lceil \frac{|y+z|}{\varepsilon} \right\rceil.$$

Suppose that $z+y = |z+y|e^{i\eta}$. Take the directions for the first N iterations of the pseudo orbit as $v_n = (0, e^{i\eta})$. In the N-th step we arrive to $(0, z+y)$. For the next iterations take the directions of the errors as $v_n = (e^{i\theta}, 0)$. Choose errors of the form $\varepsilon(e^{i\theta}, 0)$ at steps $N+1, \ldots, N+M-1$, and an error of the form $b(e^{i\theta}, 0)$ at step $N+M$, with $b = |x| - (M-1)\varepsilon$ (note that $0 < b < \varepsilon$). Then, after M iterations, we arrive to a point

$$\begin{aligned}
&T^M \begin{pmatrix} 0 \\ z+y \end{pmatrix} + \sum_{j=N+1}^{M+N-1} T^{M+N-j} \begin{pmatrix} \varepsilon e^{i\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} te^{i\theta} \\ 0 \end{pmatrix} = \\
&\begin{pmatrix} 0 \\ z+y \end{pmatrix} + \varepsilon e^{i\theta} \sum_{j=N+1}^{M+N-1} \begin{pmatrix} 1 & 0 \\ M+N-j & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} te^{i\theta} \\ 0 \end{pmatrix} = \\
&\begin{pmatrix} 0 \\ z+y \end{pmatrix} + \varepsilon e^{i\theta} \sum_{j=N+1}^{M+N-1} \begin{pmatrix} 1 \\ M+N-j \end{pmatrix} + \begin{pmatrix} te^{i\theta} \\ 0 \end{pmatrix} = \\
&\begin{pmatrix} ((M-1)\varepsilon + b)e^{i\theta} \\ z+y+\varepsilon e^{i\theta}\frac{M(M-1)}{2} \end{pmatrix} = \\
&\begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned}$$

We get that the distance between $T^M(0, z+y)$ and (x, y) is less than ε. Note that in the final step we may need to commit a smaller error to get exactly to (x, y). $\square$

So far we have that T is $(1/n^0)$-hypercyclic but not $(1/n)$-hypercyclic.

Proposition 2.2.4. If $T : \mathbb{R}^2 \to \mathbb{R}^2$, then T is $(1/n^p)$-hypercyclic for every $p < 1/2$.

Proof. As before, by Theorem 2.1.10 and Proposition 2.1.8 it suffices to show that for any $k \in \mathbb{N}$ there is a pseudo orbit starting at time k, from $(0, 0)$ to (x, y) and from (x, y) to $(0, 0)$, where (x, y) is an arbitrary vector. In order to construct a pseudo orbit from $(x, y) \in \mathbb{R}^2$ to $(0, 0)$, we can first use the errors to shrink the modulus of the first coordinate of the vectors with the goal of reaching the vertical axis. Once we arrive to the vertical axis, since every point in the vertical axis is a fixed point, we can shrink the modulus of the second coordinate until reaching $(0, 0)$.

We want to get to $(x, y) \in \mathbb{R}^2$ starting at $(0, 0)$ at the step k_0. Suppose that $x > 0$, $y < 0$ and $y = \gamma x$ with $\gamma < 0$. We repeat the same form of the pseudo orbit as before, but with errors $\varepsilon_n = 1/n^p$. This pseudo orbit will first leave the origin through the vertical axis to a vector of the form $(0, -z)$ with errors that perturb the pseudo orbit in the second coordinate during the first $N - 2$ steps after k_0. Then, only for convenience of the proof, at the step $N + k_0 - 1$ we remain in the fixed point $(0, -z)$. Then the pseudo orbit will go from $(0, z)$ to (x, y) with errors that perturb the pseudo orbit in the first coordinate during M steps. Our objective is to get a pair of points in the pseudo orbit that satisfy the following condition:

$$
\begin{aligned}
x_j <& x < x_{j+1} && (2.2.1)\\
y_j <& y && (2.2.2)
\end{aligned}
$$

The accumulation of the errors allow us to move some amount which is determined by the sum of the errors. For the sake of the proof, when this series has a decreasing general term, we can bound this series by the corresponding integrals. We obtain the following inequalities:

$$
\begin{aligned}
&\sum_{N+k_0}^{M+N+k_0-1} \frac{1}{j^p} \geq \frac{(M+N+k_0)^{1-p}}{1-p} - \frac{(N+k_0)^{1-p}}{1-p} \geq x, \qquad (2.2.3)\\
&z + \sum_{N+k_0}^{M+N+k_0-1} \frac{M+N+k_0-j-1}{j^p} \leq z\\
&\quad + \frac{M+N+k_0-1}{1-p}\left[(M+N+k_0-1)^{1-p} - (N+k_0-1)^{1-p}\right]\\
&\quad - \frac{1}{2-p}\left[(M+N+k_0-1)^{2-p} - (N+k_0-1)^{2-p}\right] = y,\\
&\sum_{k_0}^{N+k_0-2} \frac{1}{j^p} \geq \frac{(N+k_0-1)^{1-p}}{1-p} - \frac{k_0^{1-p}}{1-p} = -z.
\end{aligned}
$$

Suppose that $M = (\beta - 1)(N + k_0 - 1)$ for some $\beta > 1$. In order to have inequality (2.2.3) in terms of the expression $M + N + k_0 - 1$, we observe that for M and t big enough, there exist a constant $0 < \alpha < 1$ such that

$$(M + t)^{1-p} - t^{1-p} \geq \alpha(M + t - 1)^{1-p} - (t - 1)^{1-p}.$$

Thus, we can change (2.2.3) for

$$\alpha\left[\frac{(M+N+k_0-1)^{1-p}}{1-p}-\frac{(N+k_0-1)^{1-p}}{1-p}\right]\geq x \tag{2.2.4}$$

We get the following equation system

$$x=\alpha\frac{(N+k_0-1)^{1-p}}{1-p}(\beta^{1-p}-1) \tag{2.2.5}$$

$$y=z+\frac{(N+k_0-1)^{2-p}}{(1-p)(2-p)}(\beta^{2-p}-\beta(2-p)+1-p) \tag{2.2.6}$$

$$z=-\frac{(N+k_0-1)^{1-p}}{1-p}+\frac{k_0^{1-p}}{1-p} \tag{2.2.7}$$

Write for some $\delta>0$, $\frac{k_0^{1-p}}{1-p}=\delta x$. Denote $q(\beta)=\beta^{2-p}-\beta(2-p)+1-p$. Replacing in (2.2.6), we get

$$(\gamma-\delta)x=\frac{(N+k_0-1)^{1-p}}{1-p}\left(\frac{N+k_0-1}{2-p}q(\beta)-1\right). \tag{2.2.8}$$

Now, divide the equations (2.2.5) and (2.2.8). We get that

$$\frac{1}{\gamma-\delta}=\frac{\alpha(\beta^{1-p}-1)}{\frac{N+k_0-1}{2-p}q(\beta)-1}.$$

We solve for $N+k_0-1$ in the last equation and then replace it in (2.2.5). We have that

$$x=\alpha\frac{(2-p)^{1-p}[\alpha(\gamma-\delta)(\beta^{1-p}-1)+1]^{1-p}(\beta^{1-p}-1)}{(1-p)q(\beta)^{1-p}}:=f_{\alpha,\gamma-\delta}(\beta).$$

We analyze for which values of p, x belongs to the image of the function $f_{\alpha,\gamma-\delta}$. The following limit hold for $p<\frac{1}{2}$

$$\lim_{\beta\to 1^+}f_{\alpha,\gamma-\delta}(\beta)=+\infty.$$

Since, $\alpha(\gamma-\delta)<0$, we have that $(\frac{-1}{\alpha(\gamma-\delta)}+1)^{\frac{1}{1-p}}>1$ and

$$f_{\alpha,\gamma-\delta}\left(\left(\frac{-1}{\alpha(\gamma-\delta)}+1\right)^{\frac{1}{1-p}}\right)=0.$$

Thus, we have prove the existence of $N(\beta)\in\mathbb{R}$, $M(\beta)\in\mathbb{R}$ that solves the equations (2.2.5), (2.2.6), (2.2.7). Finally, observe that if follow the path described before with the pseudo orbit, in the step $j=\lfloor M(\beta)\rfloor+\lceil N(\beta)\rceil+k_0$, we get to a point that is down and to the left of (x,y). But, in the next iteration, in the step $j+1=\lceil M(\beta)\rceil+\lceil N(\beta)\rceil+k_0$, the pseudo orbit moves to the right. Meaning that the pseudo orbit in this step satisfies the condition (2.2.1). In order to finish the proof, we need to get exactly to the point (x,y). For this matter, we can redo the argument but instead of allowing errors $\varepsilon_n=1/n^p$, we consider $\varepsilon_n/2$. Thus, in each step of the pseudo orbit we save an extra error to correct the ending point. We get that

$$|x-x_j|<|x_{j+1}-x_j|<\frac{1}{2(j+1)^p},$$

and using the half error available, we can end the pseudo orbit at (x, y_j). Now, to correct the second coordinate, instead of going down to $(0, -z)$, we wait until the step $N + k_0$, at the fixed point $(0, -z + (y - y_j))$. Then, in the step j we would have get exactly to (x, y).

□

The unilateral shift Now we will see the first example of a (ε_n)-hypercyclic operator in a infinite dimensional space. Consider B the backward shift acting on $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$, $1 \leq p < \infty$. It is clear that B is not hypercyclic, since $\|B\| = 1$. Also, by Proposition 2.1.5, B is not (ε_n)-hypercyclic for any error sequence $(\varepsilon_n)_n \in \ell_1$. It was shown by Rolewicz, that λB is hypercyclic for every λ with $|\lambda| > 1$. We will prove that the backward shift acting on c_0 or ℓ_p, $1 \leq p < \infty$ is (ε_n)-hypercyclic for the error sequence $\varepsilon_n = \frac{1}{\sqrt{n}}$. For the proof we will need some previous results on hypercyclic families of operators.

Definition 2.2.5. Let X be a separable normed space. We say that the family of bounded operators $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ is hypercyclic if there exist some $x_0 \in X$ such that $\{T_n x_0\}_{n\in\mathbb{N}}$ is dense in X.

The following proposition gives sufficient conditions on a sequence of bounded operators for the existence of a hypercyclic vector. It is a generalization of the Hypercyclicity Criterion for a family of operators which not necessarily are the iterates of one operator [BM09, Theorem 1.6].

Proposition 2.2.6. Let X be a separable F-space, and let T_n be a sequence of linear bounded operators acting on X. Suppose that there exist a dense subset $D \subset X$ and a family $S_n : D \to X$ of maps such that

1. $T_n x \to 0$, for every $x \in D$
2. $S_n x \to 0$, for every $x \in D$
3. $T_n S_n x \to x$, for every $x \in D$

Then, the set of hypercyclic vectors of the family $\{T_n\}$ is a G_δ dense set in X.

The previous proposition will be helpful to prove that the backward shift is (ε_n)-hypercyclic for several error sequences.

Proposition 2.2.7. Let B be the backward shift acting on any of the spaces $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$, $1 \leq p < \infty$. Consider $(\alpha_n)_n$ a sequence of positive real numbers such that $(\alpha_n) \in c_0 \setminus \ell_1$. Define the error sequence $\varepsilon_n = \alpha_n \prod_{i=1}^{n-1}(1 + \alpha_i)$. Then, B is (ε_n)-hypercyclic.

Proof. Consider the sequence $\lambda_n := \prod_{i=1}^{n}(1 + \alpha_i)$. Since $(\alpha_n) \in c_0 \setminus \ell_1$, it is easy to see that $lim_{n\to\infty} \lambda_n = +\infty$. We will apply the Hypercyclicity Criterion 2.2.6 to the family $\{\lambda_n B^n\}$. Denote $D = c_{00}$ the set of finite sequences and $S_n = \frac{1}{\lambda_n} S^n$, where S is the standard forward shift. It is clear that D is dense in X and that $\lambda_n B^n x \to 0$, for every $x \in D$, since $B^n x = 0$ if $n > N$ and $x_j = 0$ for every $j > N$. Fix $x \in c_{00}$ and $\varepsilon > 0$. Suppose that $x_j = 0$ for every $j > N$. Take $n_0 \in \mathbb{N}$ such that $\|x\|_\infty < \varepsilon \lambda_n$ for all $n \geq n_0$. Then, if $n \geq n_0$, we get that

$$\|S_n x\|_p = \frac{1}{\lambda_n} \left(\sum_{j=1}^{N} |x_j|^p \right)^{1/p} = \frac{\|x\|_p}{\lambda_n} < \varepsilon.$$

By Proposition 2.2.6, the family $\{\lambda_n B^n\}$ is hypercyclic. Thus, there exists a vector x_0 such that the set $\{\lambda_n B^n x_0\}_{n\in\mathbb{N}}$ is dense in X. We can now think of this set as a dense ε_n-pseudo orbit for the operator B. Define $x_n := \lambda_n B^n x_0$, we have that

$$x_n = \prod_{i=1}^{n}(1+\alpha_i)B^n x_0 = (1+\alpha_n)Bx_{n-1} = Bx_{n-1} + \alpha_n\lambda_{n-1}B^n x_0,$$

hence, if $\varepsilon_n = \alpha_n\lambda_{n-1}$, we get that

$$\|x_n - Bx_{n-1}\| = \alpha_n\lambda_{n-1}\|B^n x_0\| \leq \varepsilon_n\|x_0\|.$$

□

Now we will study the decay of the sequence $\varepsilon_n = \alpha_n\prod_{i=1}^{n-1}(1+\alpha_i)$ for different sequences $(\alpha_n) \in c_0 \setminus \ell_1$.

Example 2.2.8.

- As our first example, suppose that $\alpha_n = 1/n$. We get that

$$\alpha_n \prod_{i=1}^{n-1}(1+\alpha_i) = \frac{1}{n}\prod_{i=1}^{n-1}\left(1+\frac{1}{i}\right) = \frac{1}{n}\prod_{i=1}^{n-1}\left(\frac{1+i}{i}\right) = \frac{n!}{n!} = 1,$$

  thus there exists a dense pseudo orbit of the backward shift associated to a constant error sequence.

- If, however, we consider $\alpha_n = 1/2n$, we get

$$\alpha_n \prod_{i=1}^{n-1}(1+\alpha_i) = \frac{1}{2n}\prod_{i=1}^{n-1}\left(\frac{1+2i}{2i}\right) = \frac{3.5.7\ldots(2n-1)}{2.4.6\ldots(2n)} = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{4^n(n!)^2}.$$

  In order to estimate the asymptotic behaviour of this sequence we quote Stirling's inequality.

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

  Finally, we obtain that

$$\frac{1}{2n}\prod_{i=1}^{n-1}\left(\frac{1+2i}{2i}\right) = \frac{(2n)!}{4^n(n!)^2} \leq \frac{e}{\pi\sqrt{2n}} = \frac{C}{\sqrt{n}}.$$

  Thus, B is $(\frac{C}{\sqrt{n}})_n$-hypercyclic. Moreover, it is worth noting, that if $(x_n)_n$ is a dense $(\frac{C}{\sqrt{n}})_n$-pseudo orbit, then $(\frac{x_n}{C})_n$ is a dense (ε_n)-pseudo orbit for B, with $\varepsilon_n = \frac{1}{\sqrt{n}}$.

- Fix $k \in \mathbb{N}$ and consider the sequence $\alpha_n = 1/kn$. We estimate the growth of $\varepsilon_n = \alpha_{n+1}\prod_{i=1}^{n}(1+\alpha_i)$:

$$\begin{aligned}\log\varepsilon_n &= \log\left(\frac{1}{k(n+1)}\right) + \sum_{i=1}^{n}\log\left(1+\frac{1}{ki}\right) \leq \log\left(\frac{1}{k(n+1)}\right) + \sum_{i=1}^{n}\frac{1}{ki}\\ &\leq \log\left(\frac{1}{k(n+1)}\right) + \frac{1}{k}(1+\log n).\end{aligned}$$

We get that

$$\varepsilon_n \le \frac{1}{k(n+1)} n^{1/k} e^{1/k}.$$

Hence, we get a dense (ε_n)-pseudo orbit for B, with error sequence $\varepsilon_n = n^{\frac{1-k}{k}} \notin \ell_{k/(k-1)}$. In particular, $\varepsilon_n \notin \ell_1$. Notably $k/(k-1)$ is a decreasing sequence that converges to 1, so when $k \to \infty$ we get a smaller error sequence.

- Suppose we take $\alpha_n = 1/(n \log n)$, for $n \ge 2$. It is easy to see that $(\alpha_n)_{n \ge 2} \notin \ell_1$. We have that

$$\begin{aligned}
\log\left(\alpha_{n+1} \prod_{i=2}^{n} (\alpha_i + 1)\right) &= \log\left(\frac{1}{(n+1)\log(n+1)}\right) + \sum_{i=2}^{n} \log\left(\frac{1}{i \log i} + 1\right) \\
&\le \log\left(\frac{1}{(n+1)\log(n+1)}\right) + \sum_{i=2}^{n} \frac{1}{i \log i} \\
&\le \log\left(\frac{1}{(n+1)\log(n+1)}\right) + \log(\log n) + 2 \\
&= \log\left(\frac{\log(n)}{(n+1)\log(n+1)}\right) + 2 \\
&\le \log\left(\frac{1}{(n+1)}\right) + 2.
\end{aligned}$$

Therefore, we obtain that the error sequence associated to this sequence satisfies,

$$\varepsilon_n = \alpha_{n+1} \prod_{i=2}^{n} (\alpha_i + 1) \le C \frac{1}{n+1}.$$

This implies that B is $(\frac{1}{n})_n$-hypercyclic.

Remark 2.2.9. It should be noted that we only needed the following assumptions on the pair (B, X):

1. there exists a dense set D such that for every $x \in D$, there exist $n \in \mathbb{N}$ such that $B^m x = 0$ if $m > n$,
2. there exists a right inverse $B \circ S = Id$, with $\|S\| \le 1$.

For example, for every generalized backward shift we can repeat the argument in Proposition 2.2.7.

Definition 2.2.10. Recall that a generalized forward shift T, in a Hilbert space H, is an isometry such that $\bigcap_{n \ge 1} Ran(T^n) = \{0\}$.

If T is a generalized forward shift and $\{\lambda_n\}_{n \in \mathbb{N}}$ is sequence of numbers such that $\lambda_n \to \infty$, then $\{\lambda_n T^{*n}\}$ satisfies the Hypercyclicity Criterion 2.2.6. Hence, we can proceed in the same way as we did for the standard backward shift.

2.3 The spectrum of a (ε_n)-hypercyclic operator

Let X be a separable normed space. In this section we analyze the spectrum of a (ε_n)-hypercyclic operator $T \in \mathcal{L}(X)$. We denote by $\sigma_p(T)$ the point spectrum, $\sigma(T)$ the spectrum and $r(T)$ the spectrum radius of T respecively.

Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is a (ε_n)-pseudo orbit of T. For every $n \in \mathbb{N}_0$ there exist unitary vectors v_n and positive real numbers α_n such that $\alpha_n \leq \varepsilon_n$ and

$$\begin{aligned} x_{n+1} &= T^{n+1}(x_0) + T^n(\alpha_1 v_1) + T^{n-1}(\alpha_2 v_2) + \cdots + T(\alpha_n v_n) + \alpha_{n+1} v_{n+1} \\ &= T^{n+1}(x_0) + \sum_{k=0}^{n} T^{n-k}(\alpha_{k+1} v_{k+1}). \end{aligned}$$

Proposition 2.3.1. Let X be a separable normed space. Suppose that $T \in \mathcal{L}(X)$ is (ε_n)-hypercyclic. Then,

- T has dense range and $\sigma_p(T^*) \subset \mathbb{S}^1$
- if $(\varepsilon_n) \in \ell_1$, then $\sigma_p(T^*) = \emptyset$.

Proof. Suppose that there exist $\varphi \in Ker(T^* - \lambda)$, $\varphi \neq 0$ with $|\lambda| < 1$. Then,

$$\begin{aligned} |\varphi(x_{n+1})| &= \left|\varphi\left(T^{n+1}(x_0) + \sum_{k=0}^{n} T^{n-k}(\alpha_{k+1} v_{k+1})\right)\right| = \left|\lambda^{n+1}\varphi(x_0) + \sum_{k=0}^{n} \alpha_{k+1}\varphi T^{n-k}(v_{k+1})\right| \\ &= \left|\lambda^{n+1}\varphi(x_0) + \sum_{k=0}^{n} \alpha_{k+1}\lambda^{n-k}\varphi(v_{k+1})\right| \leq |\lambda|^{n+1}\|\varphi(x_0)\| + \|(\varepsilon_k)\|_\infty \|\varphi\| \sum_{k=0}^{n} |\lambda|^{n-k} \\ &\leq \|\varphi(x_0)\| + C\|(\varepsilon_k)\|_\infty \|\varphi\| \frac{1}{1-|\lambda|}. \end{aligned}$$

Which means that no pseudo-orbit can be dense in X.

On other hand, suppose that there exist $\varphi \in Ker(T^* - \lambda)$, $\varphi \neq 0$ with $|\lambda| > 1$. Then,

$$\begin{aligned} |\varphi(x_{n+1})| &\geq |\varphi(T^{n+1}(x_0))| - \sum_{k=0}^{n} |\varphi(T^{n-k}(\alpha_{k+1} v_{k+1}))| \\ &\geq |\lambda|^{n+1}|\varphi(x_0)| - \|(\varepsilon_k)\|_\infty \|\varphi\| \sum_{k=0}^{n} |\lambda|^{n-k} \\ &= |\lambda|^{n+1}\left[|\varphi(x_0)| - \|(\varepsilon_k)\|_\infty \|\varphi\| \frac{|\lambda|^{n+1}-1}{|\lambda|^{n+1}(|\lambda|-1)}\right]. \end{aligned}$$

Suppose that $\{x_n\}_{n\in\mathbb{N}_0}$ is a dense pseudo orbit and that x_0 is such that

$$|\varphi(x_0)| > \|(\varepsilon_k)\|_\infty \|\varphi\| \sup_{n\in\mathbb{N}_0} \frac{|\lambda|^{n+1}-1}{|\lambda|^{n+1}(|\lambda|-1)}.$$

This can be done because $\|(\varepsilon_k)\|_\infty \|\varphi\| \frac{|\lambda|^n-1}{|\lambda|^n(|\lambda|-1)}$ is upper bounded, and if x_0 does not satisfy the previous inequality then we can shift forward the pseudo orbit and obtain a new starting point that satisfies this condition (the density of the new pseudo orbit will remain the same since only finite vectors are removed). Then, we get that $|\varphi(x_{n+1})| \nearrow +\infty$, which contradicts the fact that $(x_n)_n$ is dense. This means that no pseudo orbit can be dense.

Thus, if T is (ε_n)-hypercyclic then $\sigma_p(T^*) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Also, since $\{0\} = Ker(T^*) = R(T)^\perp$, we get that the operator T has dense range.

Moreover, if $(\varepsilon_n)_n \in \ell_1$, suppose that there exist $\varphi \in Ker(T^* - \lambda)$, $\varphi \neq 0$ with $|\lambda| \leq 1$,

$$\begin{aligned} |\varphi(x_{n+1})| &= \left|\varphi\left(T^{n+1}(x_0) + \sum_{k=0}^{n} T^{n-k}(\alpha_{k+1}v_{k+1})\right)\right| \\ &\leq \|\varphi\|\|x_0\| + \left|\sum_{k=0}^{n} \alpha_{k+1}\lambda^{n-k}\varphi(v_{k+1})\right| \\ &\leq \|\varphi\|\|x_0\| + \|(\varepsilon_k)\|_1\|\varphi\|. \end{aligned}$$

Therefore we get that $\sigma_p(T^*) = \emptyset$. □

We now focus our attention in (ε_n)-hypercyclic operators with summable error sequence. We can deduce several properties which are analogous to properties of hypercyclic operators. Recall that finite dimensional spaces do not support hypercyclic operators and that no compact operator can be hypercyclic. Also, every connected component of the spectrum of an hypercyclic operator intersects the unit disk.

Corollary 2.3.2. There are no (ε_n)-hypercyclic operators on $\mathbb{C}^N$ for any error sequence, if $(\varepsilon_n) \in \ell_1$.

The following gives further evidence of the similarities between the class of hypercyclic operators and the class of (ε_n)-hypercyclic operators with sumable error sequence.

Proposition 2.3.3. Let (ε_n) be a sumable error sequence and suppose that X is a complex F-space. Then no compact operator can be (ε_n)-hypercyclic.

Proof. Suppose that T is compact and (ε_n)-hypercyclic, with $(\varepsilon_n) \in \ell_1$. Then, the transpose of T, T^* is compact and by Proposition 2.3.1, the space X is infinite-dimensional and $\sigma_p(T^*) = \emptyset$. Thus, we get that $r(T) = 0$, since $\sigma(T) = \sigma(T^*) = \{0\}$. Which is a contradiction. □

We have proved that if the error sequence is summable then $\sigma_p(T^*) = \emptyset$. This property implies that $T - \mu$ has dense range for every $\mu \in \mathbb{C}$, and this fact implies the following proposition, which is a more general fact.

Proposition 2.3.4. Let X be a complex Banach space and T be a (ε_n)-hypercyclic with summable error sequence. If P is a non zero complex polynomial, then $P(T)$ has dense range.

Proof. We can factorize $P(T)$ as $P(T) = a_d(T - \mu_1)\dots(T - \mu_d)$. Since, $\sigma_p(T^*) = \emptyset$, we get that $(T - \mu_j)$ has dense range for every μ_j. Thus, $P(T)$ has dense range.

□

Next we state a Comparison Principle for (ε_n)-hypercyclicity.

Proposition 2.3.5. Let X and Y be two normed vector spaces, $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$ and $(\varepsilon_n)_n$ be an error sequence. Suppose there exist an operator $J : X \to Y$ with dense range such that $JT = SJ$. If T is (ε_n)-hypercyclic, then so is S.

Proof. Suppose that $(x_n)_n \subset X$ is a dense $(\varepsilon_n)_n$-pseudo orbit for T. Consider $y_n = Jx_n$, which is dense in Y since J is continuous. Also,

$$\|y_n - Sy_{n-1}\| = \|Jx_n - SJx_{n-1}\| \leq \|J\|\|x_n - Tx_{n-1}\| \leq \|J\|\varepsilon_n.$$

Thus, $(y_n/\|J\|)_n$ is dense (ε_n)-pseudo orbit for S. □

We can summarize some facts about (ε_n)-hypercyclic operators with norm 1 on finite dimensional spaces in the following theorem.

Theorem 2.3.6. Let T be an operator defined on the Euclidean space $\mathbb{C}^N$, with $\|T\| = 1$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an error sequence. The following are equivalent.

i) T is (ε_n)-hypercyclic.

ii) $(\varepsilon_n) \notin \ell_1$ and $\sigma(T^*) \subset \mathbb{S}^1$.

Proof. *i)* $\Rightarrow$ *ii)* We have seen in Proposition 2.1.5 that if T is (ε_n)-hypercyclic and $\|T\| = 1$, then $(\varepsilon_n) \notin \ell_1$. Also by Proposition 2.3.1, we get that $\sigma(T^*) = \sigma_p(T^*) \subset \mathbb{S}^1$.

ii) $\Rightarrow$ *i)* Let us first prove that T is diagonalizable. Suppose on the contrary that there exist $\lambda \in \mathbb{S}^1$ and nonzero vectors v and u such that $Tv = \lambda v$ and $Tu = \lambda u + v$. Let w be a vector on the linear span of $\{u, v\}$, orthogonal to v. Then, if $w = \alpha u + \beta v$

$$\|w\|^2 \geq \|Tw\|^2 = \|\lambda w + \alpha v\|^2 = \|w\|^2 + |\alpha|^2\|v\|^2.$$

Thus $\alpha = 0$, which is a contradiction. Therefore, by Proposition 2.3.5, we can assume that T is a diagonal map on $\mathbb{C}^N$, with eigenvalues $\lambda_1, \ldots, \lambda_N$ in $\mathbb{S}^1$. By Proposition 2.2.1, since $(\varepsilon_n) \notin \ell_1$, we get that T is (ε_n)-hypercyclic. □

2.4 Criteria for (ε_n)-hypercyclicity

In this section we provide some criteria to determine if an operator is (ε_n)-hypercyclic for a given error sequence. We will provide criteria in the spirit of Kitai's criterion as well as some eigenvalues criteria. The Hypercyclicity Criterion first appeared on the Ph.D. thesis of Kitai. The original idea was to build an orbit which, in turns, visit neighborhoods of 0 and balls with centers at a dense set, with rational radius. Simpler proofs were then given applying Baire's category theorem. Our main criterion is based on this idea of building a pseudo orbit that goes to zero, visits an open set, returns to zero, visits another open set and so on. This proposition provides us a helpful tool to check if an operator is (ε_n)-hypercyclic, when the error sequence is not p-summable, for some $p \geq 1$. An application of this proposition gives us an eigenvalue criterion for (ε_n)-hypercyclicity. Finally, we relate this concept with chaos.

Our first proposition gives sufficient conditions for (ε_n)-hypercyclicity when $(\varepsilon_n)_n \notin \ell_1$.

Proposition 2.4.1. Let T be an operator on a normed vector space X and let $(\varepsilon_n)_n \notin \ell_1$ be an error sequence. Suppose that there exist two dense sets D_1 and D_2 and a map $S : D_2 \to D_2$ such that D_2 is closed under product by scalars and $S(\lambda y) = \lambda S(y)$ for every $y \in D_2$, satisfying

1. $\{T^j x\}_{j\in\mathbb{N}}$ is bounded for every $x \in D_1$,
2. $\{S^j y\}_{j\in\mathbb{N}}$ is bounded for every $y \in D_2$,
3. $TSy = y$ for every $y \in D_2$.

Then T is (ε_n)-hypercyclic.

Proof. We will show that T satisfies condition (1) of Theorem 2.1.10. We will first see that given $x \in D_1$ and $m \in \mathbb{N}$, there exist a finite pseudo orbit for T which starts at x and ends at 0, with errors from iteration m. By hypothesis there exists $n \in \mathbb{N}$ such that $b^{-1} := \sum_{j=0}^{n} \frac{\varepsilon_{j+m}}{\|T^{j+1}x\|} > 1$. Define $x_0 = x$ and for $j = 0, \ldots, n$,

$$x_{j+1} = Tx_j - b\varepsilon_{m+j}\frac{Tx_j}{\|Tx_j\|} = Tx_j\left(1 - \frac{b\varepsilon_{m+j}}{\|Tx_j\|}\right).$$

Note that

$$x_{j+1} = T^{j+1}x\left(1 - b\sum_{l=0}^{j}\frac{\varepsilon_{l+m}}{\|T^{l+1}x\|}\right),$$

indeed proceeding by induction,

$$\begin{aligned} x_{j+1} &= Tx_j\left(1 - b\frac{\varepsilon_{m+j}}{\|Tx_j\|}\right) \\ &= T^{j+1}x\left(1 - b\sum_{l=0}^{j-1}\frac{\varepsilon_{l+m}}{\|T^{l+1}x\|}\right)\left(1 - b\frac{\varepsilon_{m+j}}{\|T^{j+1}x\|\left(1 - b\sum_{l=0}^{j-1}\frac{\varepsilon_{l+m}}{\|T^{l+1}x\|}\right)}\right) \\ &= T^{j+1}x\left(1 - b\sum_{l=0}^{j}\frac{\varepsilon_{l+m}}{\|T^{l+1}x\|}\right). \end{aligned}$$

Note that $\|x_{j+1} - Tx_j\| \leq b\varepsilon_{m+j} < \varepsilon_{m+j}$. We also have,

$$x_{n+1} = T^{n+1}x\left(1 - b\sum_{j=0}^{n}\frac{\varepsilon_{j+m}}{\|T^{j+1}x\|}\right) = 0.$$

In order to be able to have pseudo orbits starting at 0 we proceed similarly but this time using the map S. Let us see that given $y \in D_2$ and $m \in \mathbb{N}$, there exists a finite pseudo orbit for T which starts at 0, ends at y, with errors from iteration m.

Take $\alpha_j = \frac{\varepsilon_j}{\|T\|}$, and fix $k > m$ such that $c^{-1} := \sum_{j=0}^{k-m}\frac{\alpha_{j+m}}{\|S^{k-m-j+1}y\|} > 1$. Define $y_k = y$ and for $j = m, \ldots, k$,

$$y_{j-1} = Sy_j - c\alpha_j\frac{Sy_j}{\|Sy_j\|} = Sy_j\left(1 - c\frac{\alpha_j}{\|Sy_j\|}\right).$$

Then we have that

$$y_{j-1} = S^{k+1-j}y\left(1 - c\sum_{l=0}^{k-j}\frac{\alpha_{j+l}}{\|S^{k-l-j+1}y\|}\right).$$

Indeed,

$$
\begin{aligned}
y_{j-1} &= S^{k+1-j}y\left(1-c\sum_{l=0}^{k-j-1}\frac{\alpha_{j+1+l}}{\|S^{k-l-j}y\|}\right)\left(1-c\frac{\alpha_j}{\|S^{k+1-j}y\|\left(1-c\sum_{l=0}^{k-j-1}\frac{\alpha_{j+1+l}}{\|S^{k-l-j}y\|}\right)}\right)\\
&= S^{k+1-j}y\left(1-c\sum_{l=0}^{k-j}\frac{\alpha_{j+l}}{\|S^{k-l-j+1}y\|}\right).
\end{aligned}
$$

Then we have,

$$y_{m-1} = S^{k+1-m}y\left(1-t\sum_{j=1}^{k-m}\frac{\alpha_{j+m}}{\|S^{k-m-j+1}y\|}\right)=0.$$

Therefore we obtained vectors $y_{m-1}, y_m, \dots, y_k \in X$, which belong to D_2 because they are scalar multiples of vectors in the image of S, and such that $y_{m-1}=0$, $y_k=y$. Moreover,

$$\|Ty_{j-1}-y_j\| = \|T(y_{j-1}-Sy_j)\| \leq \|T\|\|y_{j-1}-Sy_j\| \leq \|T\|c\alpha_j < \varepsilon_j.$$

□

In the previous proposition we proved that an operator is (ε_n)-hypercyclic when the error sequence is not summable and the operator has bounded orbits on a dense set. We can improve this result by letting the orbits of the operator go to infinity if we require also that the error sequence is not p-summable.

Proposition 2.4.2. Let T be an operator on a normed vector space X and let $(\varepsilon_n)_n$ be a decreasing error sequence, $(\varepsilon_n)_n \notin \ell_p$ for some $p>1$. Suppose that there exist two dense sets D_1 and D_2 and a map $S: D_2 \to D_2$ such that D_2 is closed under product by scalars and $S(\lambda y)=\lambda S(y)$ for every $y \in D_2$, satisfying

1. given $x \in D_1$, there exists $C_x>0$ such that $\|T^jx\| \leq \frac{C_x}{\varepsilon_j^{p-1}}$ for every $j \in \mathbb{N}$,
2. given $y \in D_2$, there exists $D_y>0$ and $\delta>0$ such that $\|S^jy\| \leq \frac{D_y}{\varepsilon_j^{p-1-\delta}}$ for every $j \in \mathbb{N}$,
3. $TSy=y$ for every $y \in D_2$.

Then T is (ε_n)-hypercyclic.

Proof. The proof is analogous to the previous one, in fact it follows the same lines. The main difference is found when studying the series

$$\sum_{j=0}^{n}\frac{\varepsilon_{j+m}}{\|T^{j+1}x\|} \qquad \text{and} \qquad \sum_{j=1}^{k-m}\frac{\alpha_{j+m}}{\|S^{k-m-j+1}y\|}.$$

We will show that under the new hypotheses these series are also divergent. For the first one, suppose that $x \in D_1$ and $C_x>0$ are such that $\|T^jx\| \leq \frac{C_x}{\varepsilon_j^{p-1}}$ for every $j \in \mathbb{N}$. Since, the error sequence is decreasing we get that

$$\sum_{j=0}^{n}\frac{\varepsilon_{j+m}}{\|T^{j+1}x\|} \geq \sum_{j=0}^{n}\varepsilon_{j+m}\frac{\varepsilon_{j+1}^{p-1}}{C_x} \geq \frac{1}{C_x}\sum_{j=0}^{n}\varepsilon_{j+m}^{p}.$$

Thus, there exist some $n \in \mathbb{N}$ such that the sum is bigger than 1, because $(\varepsilon_n)_n \notin \ell_p$.

For the second series, suppose that $y \in D_2$ and $D_y > 0$ are such that $\|S^j y\| \leq \frac{D_y}{\varepsilon_j^{p-1}}$ for every $j \in \mathbb{N}$. We have that

$$\sum_{j=1}^{k-m} \frac{\alpha_{j+m}}{\|S^{k-m-j+1}y\|} \geq \frac{1}{D_y\|T\|} \sum_{j=1}^{k-m} \varepsilon_{j+m}\varepsilon_{k-m-j+1}^{p-1-\delta}.$$

Suppose that $\sum_{j=1}^{k-m} \varepsilon_{j+m}\varepsilon_{k-m-j+1}^{p-1-\delta} < K$ for every $k > m$. Then, since the error sequence is decreasing we get that

$$\frac{K}{k-m} > \frac{1}{k-m} \sum_{j=1}^{k-m} \varepsilon_{j+m}\varepsilon_{k-m-j+1}^{p-1-\delta} \geq \min_{1 \leq j \leq k-m} \varepsilon_{j+m}\varepsilon_{k-m-j+1}^{p-1-\delta} \geq \varepsilon_k \varepsilon_{k-m}^{p-1-\delta}$$
$$\geq \varepsilon_k^{p-\delta}.$$

Therefore, we get that $(k-m)\varepsilon_k^{p-\delta} < K$ for every $k > m$ and so

$$\varepsilon_k^p < \frac{K}{(k-m)^{\frac{p}{p-\delta}}},$$

which is a contradiction since $(\varepsilon_k) \notin \ell_p$. Thus, both series are divergent which allow us to prove the existence of finite pseudo orbits to use Theorem 2.1.10, just as we did in the previous proposition. $\square$

We continue our investigation relating (ε_n)-hypercyclic operators with chaotic operators. First we recall some definitions and suitable properties. There exists a connection between the hypercyclicity of an operator and its spectrum. Recall the Godefroy - Shapiro Criterion 1.1.18 which states that the existence of many eigenvectors ensure hypercyclicity. This theorem can be extended in the following direction.

Definition 2.4.3. Let (X, d) be a metric space and let $f : X \to X$ be a continuous function. We say that f is chaotic if f is topologically transitive and the periodic points of f are dense.

From now on, we assume that X is a separable infinite dimensional complex Banach space. In this setting, an operator $T \in \mathcal{L}(X)$ is chaotic if and only if T is hypercyclic and the periodic points of T are dense.

Proposition 2.4.4. If T admits periodic points, then there exist eigenvectors associated to eigenvalues which are a root of unity. Even more,

$$Per(T) = \text{span}\{x \in X : \text{ exist } n \in \mathbb{N} \text{ and } \lambda \in \mathbb{C} \text{ with } \lambda^n = 1 \text{ and } Tx = \lambda x\}.$$

As an immediate consequence we obtain the following generalization of Theorem **??**.

Corollary 2.4.5. Let $T \in \mathcal{L}(X)$. If

$$\bigcup_{|\lambda|>1} Ker(T-\lambda), \quad \bigcup_{|\lambda|<1} Ker(T-\lambda) \quad \text{and} \quad \bigcup_{|\lambda|=1} Ker(T-\lambda),$$

span dense subspaces, then T is chaotic.

Thus, if T has sufficiently many eigenvectors associated to eigenvalues of modulus less than one, and of modulus greater than one, must be hypercyclic. Moreover, if T also has many eigenvectors associated to eigenvalues of modulus equal than one then T is chaotic. We now show that this last condition alone implies that T is (ε_n)-hypercyclic for any $(\varepsilon_n) \notin \ell_1$.

Proposition 2.4.6. Let $T \in \mathcal{L}(X)$ and $(\varepsilon_n)_n$ be an error sequence, $(\varepsilon_n)_n \notin \ell_1$. If $\bigcup_{|\lambda|=1} Ker(T-\lambda)$ spans a dense subspace, then T is (ε_n)-hypercyclic.

Proof. We apply Proposition 2.4.1, with dense sets $D := D_1 = D_2 = \operatorname{span} \bigcup_{|\lambda|=1} Ker(T-\lambda)$. If $x \in D$, then we can write $x = \sum_{i=1}^k x_i$ for $x_i \in Ker(T-\lambda_i)$ and $|\lambda_i| = 1$, $i = 1, \dots k$. Then, for $j \in \mathbb{N}$

$$\|T^j x\| = \left\| \sum_{i=1}^k \lambda_i^j x_i \right\| \leq \sum_{i=1}^k \|x_i\| := C_x.$$

Note that D is a subspace. If $y \in D$ and $Ty = \lambda y$ for some λ with $|\lambda| = 1$, define $Sy = \frac{1}{\lambda} y$ and then extend by linearity. It is clear that S is well defined because the spaces $Ker(T-\lambda)$ are linearly independent for different values of λ. Moreover, $S(D) \subset D$ and $S(\alpha y) = \alpha S(y)$. Suppose that $y = \sum_{i=1}^k y_i \in D$, with $y_i \in Ker(T-\lambda_i)$ and $|\lambda_i| = 1$, $i = 1, \dots k$. Then, for $j \in \mathbb{N}$

$$\|S^j y\| = \left\| \sum_{i=1}^k \frac{1}{\lambda_i^j} y_i \right\| \leq \sum_{i=1}^k \|y_i\| := D_y,$$

and

$$TSy = TS\left(\sum_{i=1}^k y_i\right) = \sum_{i=1}^k TSy_i = \sum_{i=1}^k y_i = y.$$

Hence, T is (ε_n)-hypercyclic as we wanted to prove. □

2.5 Weighted shifts

2.5.1 Unilateral weighted shifts

In this section we study (ε_n)-hypercyclicity for the class of unilateral weighted backward shifts defined on $c_0(\mathbb{N})$ or $\ell_q(\mathbb{N})$, $1 \leq q < \infty$. Let $\{w_n\}_{n\in\mathbb{N}}$ be a bounded sequence of positive numbers. Let B_w be a unilateral weighted backward shift on $c_0(\mathbb{N})$ or $\ell_q(\mathbb{N})$, $1 \leq q < \infty$. Recall that B_w is the operator defined by $B_w(e_1) = 0$ and $B_w(e_n) = w_n e_{n-1}$ for $n \geq 2$. Let us denote X the space $c_0(\mathbb{N})$ or $\ell_q(\mathbb{N})$, $1 \leq q < \infty$. The following theorem is a characterization of the hypercyclicity of B_w [BM09, Proposition 1.40].

Proposition 2.5.1. Let B_w be a weighted backward shift acting on X. Then B_w is hypercyclic if and only if the sequence $(\prod_{i=1}^n w_i)_{n\in\mathbb{N}}$ is not upper bounded.

Our next result shows that for the class of unilateral weighted backward shifts, if $(\varepsilon_n) \in \ell_1$ then (ε_n)-hypercyclicity and hypercyclicity are equivalent.

Theorem 2.5.2. Let B_w be a weighted backward shift acting on X. Suppose that (ε_n) is a summable error sequence. Then, B_w is hypercyclic if and only if B_w is (ε_n)-hypercyclic.

Proof. It is clear that if B_w is hypercyclic then it is (ε_n)-hypercyclic for any error sequence. Suppose that $(x_n)_{n>0}$ is a (ε_n)-pseudo orbit for B_w. Suppose also that $(\prod_{i=1}^n w_i)_{n\in\mathbb{N}}$ is upper bounded by a positive constant M. There exist unitary vectors $v_n \in X$ and positive numbers $\alpha_n \leq \varepsilon_n$, such that for $n \geq 1$,

$$x_{n+1} = B_w^{n+1}x_0 + \sum_{k=0}^{n} \alpha_{n+1-k} B_w^k(v_{n+1-k}).$$

Denote by π_n the projection on the n-th coordinate of X. Then,

$$\begin{aligned} |\pi_1(x_{n+1})| &= \left| \left(\prod_{i=1}^{n+1} w_i\right) \pi_{n+2}(x_0) + \sum_{k=0}^{n} \alpha_{n+1-k} \left(\prod_{i=1}^{k} w_i\right) \pi_{k+1}(v_{n+1-k}) \right| \\ &\leq M(\|x_0\| + \|(\varepsilon_n)\|_{\ell_1}) < \infty. \end{aligned}$$

Thus, the pseudo orbit cannot be dense and the operator B_w cannot be (ε_n)-hypercyclic. $\square$

Remark 2.5.3. Analogously, if $(\prod_{i=1}^n w_i)_{n\in\mathbb{N}} \in \ell_r$ and $(\varepsilon_n)_n \in \ell_{r'}$ with $r^{-1} + r'^{-1} = 1$, then the operator B_w cannot be (ε_n)-hypercyclic. Just apply Hölder's inequality in the previous proof.

Proposition 2.5.4. Let B_w be a unilateral weighted backward shift acting on X and let (ε_n) be an error sequence with $(\varepsilon_n)_n \notin \ell_1$. Suppose that there exists $D > 0$ such that for every $j \in \mathbb{N}$

$$D \leq \inf_{n>j} \left| \prod_{i=n-j+1}^{n} w_i \right|.$$

Then, B_w is (ε_n)-hypercyclic.

Proof. We apply Proposition 2.4.1. Take as dense set $D := D_1 = D_2 = c_{00}$, the space of finite sequences, which is closed under scalar product. Define S as the unilateral weighted forward shift

$$Sx = \left(0, \frac{x_1}{w_2}, \frac{x_2}{w_3}, \ldots\right).$$

It is clear that S is linear, $S(D) \subset D$ and $B_w Sx = x$ for every $x \in X$. Condition (1) of Proposition 2.4.1 is satisfied because for every $x \in D$ the orbit of x under B_w is finite, meaning that we can take $C_x = \max_j \varepsilon_j^{p-1}\|B_w^j x\|$. For condition (2) we can make the following estimate,

$$\begin{aligned} \|S^j x\|_q &= \left(\sum_{n>j} \left| \frac{x_{n-j}}{\prod_{n-j+1}^{n} w_i} \right|^q \right)^{1/q} \\ &\leq \frac{1}{\inf_{n>j} \left|\prod_{n-j+1}^{n} w_i\right|} \left(\sum_{n>j} x_{n-j}^q \right)^{1/q} \\ &\leq \frac{1}{D}\|x\|_q. \end{aligned}$$

Hence, the hypothesis of the criterium are satisfied and therefore the operator B_w is (ε_n)-hypercyclic as we wanted to prove. $\square$

Remark 2.5.5. The unilateral backward shift B is a weighted shift with weights $w_n = 1$ for all $n \in \mathbb{N}$. Since, $\inf_{n>j} \left|\prod_{i=n-j+1}^{n} w_i\right| = 1$ for every $j \in \mathbb{N}$, we get that if $(\varepsilon_n)_n \notin \ell_1$, then B is (ε_n)-hypercyclic. Therefore, we get the following theorem.

Corollary 2.5.6. The unilateral backward shift B acting on X is (ε_n)-hypercyclic if and only if $(\varepsilon_n)_n \notin \ell_1$.

Proposition 2.5.7. Let B_w be a unilateral weighted backward shift acting on X and let (ε_n) be a decreasing error sequence with $(\varepsilon_n)_n \notin \ell_p$ for some $p > 1$. Suppose that there exists $\delta > 0$ such that for every $j \in \mathbb{N}$

$$\varepsilon_j^{p-1-\delta} \leq \inf_{n>j} \left| \prod_{i=n-j+1}^{n} w_i \right|.$$

Then, B_w is (ε_n)-hypercyclic.

Proof. The proof follows the same lines except that for condition (2) we can make the following estimate,

$$\begin{aligned}\|S^j x\|_q &= \left(\sum_{n>j} \left| \frac{x_{n-j}}{\prod_{n-j+1}^{n} w_i} \right|^q \right)^{1/q} \\ &\leq \frac{1}{\inf_{n>j} \left| \prod_{n-j+1}^{n} w_i \right|} \left(\sum_{n>j} x_{n-j}^q \right)^{1/q} \\ &\leq \frac{1}{\varepsilon_j^{p-1-\delta}} \|x\|_q.\end{aligned}$$

Hence, the hypothesis of the Proposition 2.4.2 are satisfied and therefore the operator B_w is (ε_n)-hypercyclic as we wanted to prove. □

Corollary 2.5.8. Let w_n be a bounded sequence of positive numbers such that $0 < w_i \uparrow 1$. Suppose that there exist $p > 1$ and $\delta > 0$ such that for all $n \in \mathbb{N}$

$$w_1 \ldots w_n > \left(\frac{1}{n}\right)^{\frac{1}{p'} - \delta}.$$

Then, B_w is $(n^{-1/p})$-hypercyclic.

Proof. Just note that if $n \geq j + 1$ then,

$$w_{n-j+1} w_{n-j+2} \ldots w_n \geq w_1 w_2 \ldots w_j > \left(\frac{1}{j}\right)^{\frac{1}{p'} - \delta} = \varepsilon_j^{p-1-\delta p}.$$

Since, $(n^{-1/p})_n$ is decreasing and not p-summable, we can apply the previous proposition. □

Remark 2.5.9. Note that under this hypothesis the weighted backward shift is $(n^{-1/p})$-hypercyclic and is not hypercyclic. Thus, we can distinguish between different classes of chain transitivity in terms of the summability of the error sequence.

2.5.2 Bilateral weighted shifts

Contrary to the case of unilateral weighted shifts, for bilateral weighted shifts the equivalence between hypercyclicity and (ε_n)-hypercyclicity with summable error sequence does not hold. A weight sequence will be a bounded sequence of positive numbers, $\{w_n\}_{n\in\mathbb{Z}}$. A bilateral weighted backward shifts defined on $c_0(\mathbb{Z})$ or $\ell_q(\mathbb{Z})$, $1 \le q < \infty$, B_w, is the operator defined by $B_w(e_n) = w_n e_{n-1}$ for $n \in \mathbb{Z}$. Let us denote X the space $c_0(\mathbb{Z})$ or $\ell_q(\mathbb{Z})$, $1 \le q < \infty$. The following theorem is a characterization of the hypercyclicity of B_w in terms of the weight sequence [BM09, Proposition 1.39].

Theorem 2.5.10. Let B_w be a bilateral weighted backward shift acting on X. Then B_w is hypercyclic if and only if, for any $l \in \mathbb{N}$,

$$\liminf_{n\to+\infty} \max\{(w_1 \cdots w_{n+l})^{-1}, (w_0 \cdots w_{-n+l+1})\} = 0.$$

When the shift B_w is invertible this condition can be replaced by the following equivalent statement. The operator B_w is hypercyclic if and only if,

$$\liminf_{n\to+\infty} \max\{(w_1 \cdots w_n)^{-1}, (w_{-1} \cdots w_{-n})\} = 0.$$

Since, the weights w_n are bounded above and below, the corresponding products are equivalent, up to constants depending only on l.

Surprisingly the behavior of bilateral weighted shifts is rather different from unilateral weighted shifts. Our next example shows that (ε_n)-hypercyclicity with $(\varepsilon_n) \in \ell_1$ is not equivalent to hypercyclicity.

Example 2.5.11. Let us consider the bilateral weighted shift acting on X defined by the following weight sequence

$$(w_n) = \left(\ldots, 2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, \frac{1}{2}, \underbrace{1}_{n=0}, \frac{1}{2}, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2, 2, 2, \ldots \right)$$

It is clear that B_w is invertible. Also it is easy to see that B_w is not hypercyclic. Since $w_j = w_{-j}$ for every $j \in \mathbb{N}$, we get that $w_1 \cdots w_n = w_{-1} \cdots w_{-n}$. Thus,

$$\max\{(w_1 \cdots w_n)^{-1}, (w_{-1} \cdots w_{-n})\} \ge 1.$$

The next observations will be usefull. For $m \in \mathbb{N}$ and $j \in \mathbb{Z}$, simple calculation shows that,

$$B_w^m(x)(j) = \left(\prod_{i=1}^{m} w_{j+i}\right) x(j+m) \quad \text{and} \quad B_w^{-m}(x)(j) = \left(\prod_{i=1}^{m} w_{j-i+1}\right)^{-1} x(j-m).$$

Note that

$$\prod_{i=1}^{\frac{k(k+1)}{2}} w_i = \begin{cases} 2^{\frac{k}{2}} & \text{if } k \text{ is even} \\ \left(\frac{1}{2}\right)^{\frac{k-1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

We will prove that B_w is $(1/n^2)$-hypercyclic. For that by Theorem 2.1.10, it is sufficient to prove the following two facts: $i)$ given $x \in c_{00}(\mathbb{Z})$ and $j \in \mathbb{N}$ there exists a finite pseudo orbit starting at x and ending at 0 with errors from iteration j,
$ii)$ given $x \in c_{00}(\mathbb{Z})$ and $j \in \mathbb{N}$ there exists $y_j \in B(0, \varepsilon_j)$ and a finite pseudo orbit starting at y_j and ending at x with errors from iteration j.

For $i)$, fix $x \in c_{00}(\mathbb{Z})$ and $j \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $x_n = 0$ for all n with $|n| > N$ and let $k \in \mathbb{N}$ even. We get that

$$\begin{aligned}
\left\| B_w^{\frac{(k+1)(k+2)}{2}} x \right\|_X &\leq \sum_{l=-N}^{N} \left| B_w^{\frac{(k+1)(k+2)}{2}} x \left(-\frac{(k+1)(k+2)}{2} + l \right) \right| \\
&= \sum_{l=-N}^{N} |x(l)| \left| \prod_{i=-\frac{(k+1)(k+2)}{2}+l+1}^{l} w_i \right| \\
&\leq \sum_{l=-N}^{N} |x(l)| 2^{2|l|+1} \left| \prod_{i=-\frac{(k+1)(k+2)}{2}}^{-1} w_i \right| \\
&\leq (2N+1) 2^{2N+1} \left(\frac{1}{2} \right)^{\frac{k}{2}} \|x\|_\infty
\end{aligned}$$

Now, we let k be even such that

$$(2N+1) 2^{2N+1} \left(\frac{1}{2} \right)^{\frac{k}{2}} \|x\|_\infty < \left(\frac{1}{\frac{(k+1)(k+2)}{2} + j} \right)^2 = \varepsilon_{\frac{(k+1)(k+2)}{2}+j}.$$

Thus, $\{x, B_w x, \ldots, B_w^{\frac{(k+1)(k+2)}{2}-1} x, 0\}$ is a pseudo orbit from x to 0 with errors starting at ε_j.

For $ii)$, fix $x \in c_{00}(\mathbb{Z})$ and $j \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $x_n = 0$ for all n with $|n| > N$. We get that, for k even,

$$\begin{aligned}
\left\| B_w^{-\frac{k(k+1)}{2}} x \right\|_X &\leq \sum_{l=-N}^{N} \left| B_w^{-\frac{k(k+1)}{2}} x \left(\frac{k(k+1)}{2} + l \right) \right| \\
&= \sum_{l=-N}^{N} |x(l)| \left| \prod_{i=l+1}^{\frac{k(k+1)}{2}+l} w_i \right|^{-1} \\
&\leq \sum_{l=-N}^{N} |x(l)| 2^{2|l|+1} \left| \prod_{i=1}^{\frac{k(k+1)}{2}} w_i \right|^{-1} \\
&\leq (2N+1) 2^{2N+1} \left(\frac{1}{2} \right)^{\frac{k}{2}} \|x\|_\infty
\end{aligned}$$

Now, we let k be even such that

$$(2N+1) 2^{2N+1} \left(\frac{1}{2} \right)^{\frac{k}{2}} \|x\|_\infty < \left(\frac{1}{j} \right)^2 = \varepsilon_j.$$

Thus, the orbit starting at $B_w^{-\frac{k(k+1)}{2}} x \in B(0, \varepsilon_j)$ ends at x in $\frac{k(k+1)}{2}$ iterations of B_w. Therefore, by Theorem 2.1.10, B_w is $(1/n^2)$-hypercyclic.

Chapter 3

Hypercyclic behavior of some non-convolution operators on $H(\mathbb{C}^N)$

We study hypercyclicity properties of a family of non-convolution operators defined on spaces of holomorphic functions on $\mathbb{C}^N$. These operators are a composition of a differentiation operator and an affine composition operator, and are analogues of operators studied by Aron and Markose on $H(\mathbb{C})$. The hypercyclic behavior is more involved than in the one dimensional case, and depends on several parameters involved.

Introduction

The first examples of hypercyclic operators were found by Birkhoff [Bir29] and MacLane [Mac52], whose research was focused in holomorphic functions of one complex variable and not in properties of operators. Birkhoff's result implies that the translation operator $\tau : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $\tau(h)(z) = h(1 + z)$ is hypercyclic. Likewise, MacLane's result states that the differentiation operator on $H(\mathbb{C})$ is hypercyclic. In a seminal paper, Godefroy and Shapiro [GS91] unified and generalized both results, by showing that every continuous linear operator $T : H(\mathbb{C}^N) \to H(\mathbb{C}^N)$ which commutes with translations and which is not a multiple of the identity is hypercyclic. This operators are called non-trivial convolution operators.

Another important class of operators on $H(\mathbb{C}^N)$ are the composition operators C_ϕ, induced by symbols ϕ which are automorphisms of $\mathbb{C}^N$. The hypercyclicity of composition operators induced by affine automorphisms was completely characterized in terms of properties of the symbol by Bernal-González [BG05].

Besides operators belonging to some of these two classes, there are not many examples of hypercyclic operators on $H(\mathbb{C}^N)$. Motivated by this fact, Aron and Markose [AM04] studied the hypercyclicity of the following operator on $H(\mathbb{C})$, $Tf(z) = f'(\lambda z + b)$, with $\lambda, b \in \mathbb{C}$. The operator T is not a convolution operator unless $\lambda = 1$. They showed that T is hypercyclic for any $|\lambda| \geq 1$ (a gap in the proof was corrected in [FH05]) and that it is not hypercyclic if $|\lambda| < 1$ and $b = 0$. Thus, they gave explicit examples of hypercyclic operators which are neither convolution operators nor composition operators. Recently, this operators were studied in [GMar], where the authors showed that the operator is frequently hypercyclic when $b = 0$, $|\lambda| \geq 1$ and asked whether it is frequently hypercyclic for any b. In section 3.1, we give a different proof of the result of [AM04, FH05], but for any $\lambda, b \in \mathbb{C}$. We conclude in Proposition 3.1.3 that T is hypercyclic if and only if $|\lambda| \geq 1$, and that in this case, T is even strongly mixing with respect to some Borel probability measure of full support on $H(\mathbb{C})$, and in particular frequently hypercyclic.

In Section 3.2 we define N-dimensional analogues of the operators considered by Aron and Markose and study the dynamics they induce in $H(\mathbb{C}^N)$. These operators are a composition between a partial differentiation operator and a composition operator induced by some automorphism of $\mathbb{C}^N$. It turns out that its behavior is more complicated than its one variable analogue. One possible reason is that, while the automorphisms of $\mathbb{C}$ have a very simple structure and hypercyclicity properties, the automorphisms of $\mathbb{C}^N$ are much more involved. Even, the characterization of hypercyclic affine automorphisms is nontrivial (see [BG05]).

First, we consider the case in which the composition operators are induced by a diagonal operator plus a translation, that is, for $f \in H(\mathbb{C}^N)$ and $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, we study operators of the form $Tf(z) = D^\alpha f((\lambda_1 z_1, \ldots, \lambda_N z_N) + b)$, where α is a multi-index and b and $\lambda = (\lambda_1, \ldots, \lambda_N)$ are vectors in $\mathbb{C}^N$. In this case we completely characterize the hypercyclicity of these non-convolution operators in terms of the parameters involved, which contrary to the one dimensional case studied in [AM04], does not only depend on the size of λ. Finally, we study the operators which are a composition of a directional differentiation operator with a general affine automorphism of $\mathbb{C}^N$ and determine its hypercyclicity in some cases.

3.1 Non-convolution operators on $H(\mathbb{C})$

Let us denote by D and τ_a the derivation and translation operators on $H(\mathbb{C})$, respectively. Namely, for an entire function f, we have

$$D(f)(z) = f'(z) \quad \text{and} \quad \tau_a(f)(z) = f(z+a).$$

MacLane's theorem [Mac52] says that D is a hypercyclic operator, and Birkhoff's theorem [Bir29] states that τ_a is hypercyclic provided that $a \neq 0$. The translation operators is a special class of composition operators on $H(\mathbb{C})$. By a composition operator we mean an operator C_ϕ such that $C_\phi(f) = f \circ \phi$, where ϕ is some automorphism of $\mathbb{C}$. The hypercyclicity of the composition operators on $H(\mathbb{C})$ has been completely characterized in terms of properties of the symbol function ϕ. Precisely, the relevant property of ϕ is the following.

Definition 3.1.1. A sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of holomorphic maps on $\mathbb{C}$, is called *runaway* if, for each compact set $K \subset \mathbb{C}$, there is an integer $n \in \mathbb{N}$ such that $\phi_n(K) \cap K = \emptyset$. In the case where $\phi_n = \phi^n$ for every $n \in \mathbb{N}$, we will just say that ϕ is runaway.

This definition was first given by Bernal González and Montes-Rodríguez in [BGMR95], where they also proved the following (see also [GEPM11, Therorem 4.32]).

Theorem 3.1.2. *Let ϕ be an automorphism of $\mathbb{C}$. Then C_ϕ is hypercyclic if and only if ϕ is runaway.*

It is known that the automorphisms of $\mathbb{C}$ are given by $\phi(z) = \lambda z + b$, with $\lambda \neq 0$ and $b \in \mathbb{C}$. In addition, ϕ is runaway if and only if $\lambda = 1$ and $b \neq 0$ (see [GEPM11, Example 4.28]). This means that the hypercyclic composition operators on $H(\mathbb{C})$ are exactly Birkhoff's translation operators.

Aron and Markose in [AM04] studied the hypercyclicity of the following operator on $H(\mathbb{C})$,

$$Tf(z) = f'(\lambda z + b),$$

with $\lambda, b \in \mathbb{C}$, which is a composition of MacLane's derivation operator and a composition operator, i.e., $T = C_\phi \circ D$ with $\phi(z) = \lambda z + b$. The main motivation for the study of this operator was the wish to understand the behavior of a concrete operator belonging neither to the class of convolution operators nor to the class of composition operators. As mentioned before, in [AM04] (see also [FH05]) the authors proved that T is hypercyclic if $|\lambda| \geq 1$, and that it is not hypercyclic if $|\lambda| < 1$ and $b = 0$.

In this section we give a simple proof of the result by Aron and Markose, for the full range on λ, b. This will allow us to illustrate some of the main ideas used in the next section to prove the more involved N-variables case.

Suppose that $\lambda \neq 1$. The key observation is that T is conjugate to an operator of the same type, but with $b = 0$. Indeed, define $T_0 f(z) = f'(\lambda z)$, then we have that the following diagram commutes.

$$\begin{tikzcd}
H(\mathbb{C}) \arrow[r, "T"] \arrow[d, "\tau_{[\frac{b}{1-\lambda}]}"'] & H(\mathbb{C}) \\
H(\mathbb{C}) \arrow[r, "T_0"'] & H(\mathbb{C}) \arrow[u, "\tau_{[\frac{b}{\lambda-1}]}"']
\end{tikzcd}$$

Note that $\frac{b}{1-\lambda}$ is the fixed point of ϕ. This observation will be important later.

Proposition 3.1.3. *Let T be the operator defined on $H(\mathbb{C})$ by $Tf(z) = f'(\lambda z + b)$. Then T is hypercyclic if and only if $|\lambda| \geq 1$. In this case, T is also strongly mixing with respect to some Borel probability measure of full support on $H(\mathbb{C})$.*

Proof. If $\lambda = 1$, then T is a non-trivial convolution operator, thus it is hypercyclic. Moreover, by the Godefroy and Shapiro's theorem and its extensions (see [GS91, BGE06, MPS14]), T is strongly mixing in the gaussian sense. Hence, by Proposition 1.3.11, it suffices to prove the case $b = 0$ and $\lambda \neq 1$, i.e. for the operator T_0.

Suppose first that $|\lambda| < 1$ and let $f \in H(\mathbb{C})$. Note that $T_0^n f(z) = \lambda^{\frac{n(n-1)}{2}} f^{(n)}(\lambda^n z)$. By the Cauchy's estimates we obtain that

$$|T_0^n f(0)| \leq |\lambda|^{\frac{n(n-1)}{2}} n! \sup_{\|z\| \leq 1} |f(z)| \underset{n\to\infty}{\longrightarrow} 0.$$

Since the evaluation at 0 is continuous, the orbit of f under T_0 can not be dense.

Suppose now that $|\lambda| > 1$. Let us see that we can apply the Murillo-Arcila and Peris criterion, Theorem 1.3.10. Let X_0 be the set of all polynomials, which is dense in $H(\mathbb{C})$. Then, for each polynomial $f \in X_0$, the series $\sum_n T_0^n f$ is actually a finite sum, thus it is unconditionally convergent.

For $n \in \mathbb{N}$ we define a sequence of linear maps $S_n : X_0 \to X$ as

$$S_n(z^k) = \frac{k!}{(k+n)!} \frac{z^{k+n}}{\lambda^{nk + \frac{n(n-1)}{2}}}.$$

It is easy to see that S_n satisfy the hypothesis of Theorem 1.3.10.

- $T_0 \circ S_1 = I$:
$$T_0 \circ S_1(z^k) = T_0\left(\frac{1}{k+1}\frac{z^{k+1}}{\lambda^k}\right) = z^k.$$
- $T_0 \circ S_n = S_{n-1}$:
$$\begin{aligned} T_0 \circ S_n(z^k) &= T_0\left(\frac{k!}{(k+n)!}\frac{z^{k+n}}{\lambda^{nk+\frac{n(n-1)}{2}}}\right) \\ &= \frac{k!}{(k+n-1)!}\frac{\lambda^{k+n-1}z^{k+n-1}}{\lambda^{nk+\frac{n(n-1)}{2}}} \\ &= \frac{k!}{(k+n-1)!}\frac{z^{k+n-1}}{\lambda^{(n-1)k+\frac{(n-1)(n-2)}{2}}} \\ &= S_{n-1}(z^k). \end{aligned}$$
- The series $\sum_n S_n(f)$ is unconditionally convergent for each $f \in X_0$. If $|z| \leq R$, we get that,
$$\sum_n |S_n(z^k)| \leq \sum_n \frac{k!}{(k+n)!} R^{k+n} \leq k! e^R.$$

Thus, the operator T_0 is strongly mixing in the gaussian sense. $\square$

We can summarize the results of this section in the following table. It is worth noticing that nor the hypercyclicity of C_ϕ nor the hypercyclicity of D imply the hypercyclicity of $C_\phi \circ D$.

	$\lambda < 1$	$\lambda = 1$	$\lambda > 1$
C_ϕ	Not Hypercyclic	Hypercyclic $\Leftrightarrow b \neq 0$	Not Hypercyclic
D	Hypercyclic	Hypercyclic	Hypercyclic
$C_\phi \circ D$	Not Hypercyclic	Hypercyclic	Hypercyclic

3.2 Non-convolution operators on $H(\mathbb{C}^N)$

3.2.1 The diagonal case

The operators considered in the previous section were differentiation operators followed by a composition operator. In this section we consider N-dimensional analogues of those operators. First, we will be concerned with symbols $\phi : \mathbb{C}^N \to \mathbb{C}^N$, which are diagonal affine automorphisms of the form
$$\phi(z) = \lambda z + b = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N),$$
where $\lambda, b \in \mathbb{C}^N$; and the differentiation operator is a partial derivative operator given by a multi-index $\alpha = (\alpha_1 \ldots, \alpha_N) \in \mathbb{N}_0^N$,
$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \ldots \partial z_N^{\alpha_N}}.$$
Thus, in this section T will denote the operator on $H(\mathbb{C}^N)$ defined by
$$Tf(z) = C_\phi \circ D^\alpha(f)(z) = D^\alpha f(\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N).$$

Note that, in the definition of T, we allow α to be zero. In this case, the operator is just a composition operator and its hypercyclicity is determined by the symbol ϕ. These symbol functions are special cases of affine automorphisms of $\mathbb{C}^N$. The existence of universal functions for composition operators with affine symbol on $\mathbb{C}^N$ has been completely characterized by Bernal-Gonzalez in [BG05], where he proved that the hypercyclicity of the composition operator depends on whether or not the symbol is runaway. Recall that an automorphism φ of $\mathbb{C}^N$ is said to be runaway if for any compact subset K there is some $n \geq 1$ such that $\varphi^n(K) \cap K = \emptyset$.

Theorem 3.2.1 (Bernal-González)**.** *Assume that $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ is an affine automorphism of $\mathbb{C}^N$, say $\varphi(z) = Az + b$. Then, the composition operator C_φ is hypercyclic if and only if φ is a runaway automorphism, if and only if the vector b is not in $ran(A - I)$ and $det(A) \neq 0$.*

The proof of this result is based on the following N-variables generalization of Runge's approximation theorem, which will be useful for us later.

Theorem 3.2.2. *If K and L are disjoint convex compact sets in $\mathbb{C}^N$ and f is a holomorphic function in a neighborhood of $K \cup L$, then there is a sequence of polynomials on $\mathbb{C}^N$ that approximate f uniformly on $K \cup L$.*

Remark 3.2.3. It is easy to prove that the mapping $\phi(z) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$ is runaway if and only if some coordinate is a translation, that is, for some $i = 1, \ldots, N$ we have, simultaneously, that $\lambda_i = 1$ and $b_i \neq 0$.

If $\lambda_j = 0$ for some j, then we have that the differential $d(T^n f)(e_j) = 1$, for every $n \in \mathbb{N}$. Since, the application $d(\cdot)(e_j)$ is continuous, we conclude that the orbit of f under T can not be dense.

The next result completely characterizes the hypercyclicity of the operator $Tf = C_\phi \circ D^\alpha f$, with $\lambda \neq 0$ and $\alpha \neq 0$ (the case $\alpha = 0$ is covered in [BG05], and as mentioned above T is not hypercyclic if $\lambda_j = 0$ for some j). Write $\lambda^\alpha = \prod_{i \leq N} \lambda_i^{\alpha_i}$.

Theorem 3.2.4. *Let T be the operator on $H(\mathbb{C}^N)$, defined by $Tf(z) = C_\phi \circ D^\alpha f(z)$, where $\alpha \neq 0$, $\phi(z) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$ and $\lambda_i \neq 0$ for all i, $1 \leq i \leq N$. Then,*

a) If $|\lambda^\alpha| \geq 1$ then T is strongly mixing in the gaussian sense.

b) If for some $i = 1, \ldots, N$ we have that $b_i \neq 0$ and $\lambda_i = 1$, then T is mixing.

c) In any other case, T is not hypercyclic.

Remark 3.2.5. The item *c)* above includes the following cases:

$c - i)$ $|\lambda^\alpha| < 1$ and $b = 0$.

$c - ii)$ $|\lambda^\alpha| < 1$ and $\lambda_i \neq 1$ for every i, $1 \leq i \leq N$.

$c - iii)$ $|\lambda^\alpha| < 1$ and $b_i = 0$ for every i such that $\lambda_i = 1$.

In all three cases we have that the application $\phi(z) = \lambda z + b$ has a fixed point and thus ϕ is not runaway. Also in case b) the application ϕ has one coordinate which is a translation, thus it is runaway. So, in particular, Theorem 3.2.4 implies that $T = C_\phi \circ D^\alpha$ is hypercyclic if and only if either $|\lambda^\alpha| \geq 1$ or ϕ is runaway.

We can summarize our main theorem in the following table.

	$\vert\lambda^\alpha\vert < 1$ and no coord. of ϕ is a translation	$\vert\lambda^\alpha\vert < 1$ and a coord. of ϕ is a translation	$\vert\lambda^\alpha\vert \geq 1$
C_ϕ	Not Hypercyclic	Hypercyclic	depends on ϕ
D^α	Hypercyclic	Hypercyclic	Hypercyclic
$C_\phi \circ D^\alpha$	Not Hypercyclic	Hypercyclic	Hypercyclic

We will divide the proof of part (a) of Theorem 3.2.4 in two lemmas. Through a change in the order of the variables, we may suppose that the first j variables, $0 \leq j \leq N$, correspond to the coordinates in which $\lambda_i = 1$. The operator T is then of the form

$$Tf(z) = D^\alpha f(z_1 + b_1, \dots, z_j + b_j, \lambda_{j+1}z_{j+1} + b_{j+1}, \dots, \lambda_N z_N + b_N). \tag{3.2.1}$$

Moreover, we can assume that $b_i = 0$ for all $i > j$, because T is topologically conjugate to

$$T_0 f(z) = D^\alpha f(z_1 + b_1, \dots, z_j + b_j, \lambda_{j+1}z_{j+1}, \dots, \lambda_N z_N). \tag{3.2.2}$$

through a translation. Indeed, defining $c \in \mathbb{C}^N$ by $c_l = 0$ if $l \leq j$, and $c_l = \frac{b_l}{1-\lambda_l}$ if $l > j$, we get that $T_0 \circ \tau_c = \tau_c \circ T$.

We first study the case in which for some i, we have $\lambda_i \neq 1$ and $\alpha_i \neq 0$ (note that if all $\lambda_i = 1$, then T is a convolution operator and it is thus strongly mixing in the gaussian sense [BGE06, MPS14]).

Lemma 3.2.6. *Let T be as in (3.2.1). Suppose that $|\lambda^\alpha| \geq 1$ and $\alpha_i \neq 0$ for some $i > j$. Then T is strongly mixing in the gaussian sense.*

Proof. By the above comments, we may suppose that $b_i = 0$ for $i > j$, so the operator T is as in (3.2.2). We apply Theorem 1.3.10 with

$$X_0 = span\left\{e_\gamma z^\beta := e^{\gamma_1 z_1 + \dots + \gamma_j z_j} z^\beta \text{ with } \beta_i = 0 \text{ for } i \leq j \text{ and } \gamma \in \mathbb{C}^j\right\}.$$

The set $X_0 \subset H(\mathbb{C}^N)$ is dense. Indeed, since the set $\{e_\gamma : \gamma \in \mathbb{C}^j\}$ generates a dense subspace in $H(\mathbb{C}^j)$ (see for example [BGE06, Proposition 2.4]), given a monomial $z_1^{\theta_1} \dots z_j^{\theta_j}$, $\epsilon > 0$ and $R > 0$, there is $f \in span\{e_\gamma : \gamma \in \mathbb{C}^j\}$ with

$$\sup_{\|z\| \leq R} \left| f(z_1, \dots, z_j) - z_1^{\theta_1} \dots z_j^{\theta_j} \right| < \epsilon.$$

We obtain

$$\sup_{\|z\| \leq R} \left| \underbrace{f(z_1, \dots, z_j) z_{j+1}^{\beta_{j+1}} \dots z_N^{\beta_N}}_{\in X_0} - z_1^{\theta_1} \dots z_j^{\theta_j} z_{j+1}^{\beta_{j+1}} \dots z_N^{\beta_N} \right| < \epsilon R^{|\beta|}.$$

Therefore we can approximate any monomial in $H(\mathbb{C}^N)$ by functions of X_0 uniformly on compacts sets.

The series $\sum_n T^n(e_\gamma z^\beta)$ is unconditionally convergent because the operator T differentiates in some variable z_i with $i > j$, and so it is a finite sum. On the other hand, if we denote by $\alpha_{(1)} := (\alpha_1, \dots, \alpha_j)$ and $\alpha_{(2)} := (\alpha_{j+1}, \dots, \alpha_N) \neq 0$, we obtain

$$T^n(e_\gamma z^\beta) = \gamma^{n\alpha_{(1)}} e^{n\langle\gamma, b\rangle} \lambda^{n\beta - \frac{n(n+1)}{2}\alpha_{(2)}} \frac{\beta!}{(\beta - n\alpha_{(2)})!} e_\gamma z^{\beta - n\alpha_{(2)}}.$$

Now, we define a sequence of maps $S_n : X_0 \to X_0$. First, we do that on the set $\{e_\gamma z^\beta\}$ and then extending them by linearity

$$S_n(e_\gamma z^\beta) = \frac{\beta!}{\gamma^{n\alpha_{(1)}} e^{n\langle\gamma,b\rangle} \lambda^{n\beta+\frac{n(n-1)}{2}\alpha_{(2)}} (\beta+n\alpha_{(2)})!} e_\gamma z^{\beta+n\alpha_{(2)}}.$$

The following assertions hold:

- $T \circ S_1 = I$:

$$\begin{aligned} T \circ S_1(e_\gamma z^\beta) &= \frac{1}{\gamma^{\alpha_{(1)}} e^{\langle\gamma,b\rangle} \lambda^\beta} \frac{\beta!}{(\beta+\alpha_{(2)})!} T(e_\gamma z^{\beta+\alpha_{(2)}}) \\ &= \frac{1}{\gamma^{\alpha_{(1)}} e^{\langle\gamma,b\rangle} \lambda^\beta} \frac{\beta!}{(\beta+\alpha_{(2)})!} \gamma^{\alpha_{(1)}} e^{\langle\gamma,b\rangle} e_\gamma \frac{(\beta+\alpha_{(2)})!}{\beta!} z^\beta \lambda^\beta \\ &= e_\gamma z^\beta. \end{aligned}$$

- $T \circ S_n = S_{n-1}$:

$$\begin{aligned} T \circ S_n(e_\gamma z^\beta) &= \frac{1}{\gamma^{n\alpha_{(1)}} e^{n\langle\gamma,b\rangle} \lambda^{n\beta+\frac{n(n-1)}{2}\alpha_{(2)}}} \frac{\beta!}{(\beta+n\alpha_{(2)})!} T(e_\gamma z^{\beta+n\alpha_{(2)}}) \\ &= \frac{\beta! \gamma^{\alpha_{(1)}} e^{\langle\gamma,b\rangle} \lambda^{\beta+(n-1)\alpha_{(2)}} (\beta+n\alpha_{(2)})!}{\gamma^{n\alpha_{(1)}} e^{n\langle\gamma,b\rangle} \lambda^{n\beta+\frac{n(n-1)}{2}\alpha_{(2)}} (\beta+n\alpha_{(2)})! (\beta+(n-1)\alpha_{(2)})!} e_\gamma z^{\beta+(n-1)\alpha_{(2)}} \\ &= \frac{\beta!}{\gamma^{(n-1)\alpha_{(1)}} e^{(n-1)\langle\gamma,b\rangle} \lambda^{(n-1)\beta+\frac{(n-1)(n-2)}{2}\alpha_{(2)}} (\beta+(n-1)\alpha_{(2)})!} e_\gamma z^{\beta+(n-1)\alpha_{(2)}} \\ &= S_{n-1}(e_\gamma z^\beta). \end{aligned}$$

- Given $R > 0$, let $|z| \le R$ and denote $C = |\frac{R^{\alpha_{(2)}}}{\lambda^\beta \gamma^{\alpha_{(1)}} e^{\langle\gamma,b\rangle}}|$. We have $|S_n(e_\gamma z^\beta)| \le M \frac{C^n}{(\beta+n\alpha_{(2)})!}$ for some constant $M > 0$ not depending on n. Since, $\alpha_{(2)} \ne 0$, we get that for each $\gamma \in \mathbb{C}^j$ and $\beta \in \mathbb{C}^N$ with $\beta_i = 0$ for $i \le j$, $\sum_n |S_n(e_\gamma z^\beta)|$ is uniformly convergent on compacts sets.

We have thus shown that the hypothesis of Theorem 1.3.10 are fulfilled. Hence T is strongly mixing in the gaussian sense, as we wanted to prove. $\square$

The other case we need to prove is when T does not differentiate in the variables z_i with $i > j$. This means that $\alpha_i = 0$ for all $i > j$. To prove this case we will use Theorem 1.3.9.

Lemma 3.2.7. *Let T be as in (3.2.1). Suppose that $|\lambda^\alpha| \ge 1$ and $\alpha_i = 0$ for every $i > j$. Then T is strongly mixing in the gaussian sense.*

Proof. We may suppose that $b_i = 0$ for $i > j$, so the operator T is as in (3.2.2). The functions $e_\gamma z^\beta$, with $\gamma_i = 0$ for all $i > j$ and $\beta_i = 0$ for every $i \le j$, are eigenfunctions of T. Indeed,

$$T(e_\gamma z^\beta) = \gamma^{\alpha_{(1)}} e^{\sum \gamma_i(z_i+b_i)} (\lambda z)^\beta = \gamma^{\alpha_{(1)}} \lambda^\beta e^{\langle\gamma,b\rangle} e_\gamma z^\beta,$$

where, as in the proof of the last lemma, $\alpha_{(1)} = (\alpha_1, \ldots, \alpha_j) \ne 0$ (note that in this case $\alpha_{(2)} = (\alpha_{j+1}, \ldots, \alpha_N) = 0$).

By Theorem 1.3.9 it is enough to show that for every set $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the set

$$\Big\{ e_\gamma z^\beta;\ \beta \in \mathbb{C}^N \text{ with } \beta_i = 0 \text{ for } i \leq j \text{ and } \gamma_i = 0 \text{ for } i > j, \text{ such that } \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle} \in \mathbb{T} \setminus D \Big\} \tag{3.2.3}$$

spans a dense subspace on $H(\mathbb{C}^N)$.

Fix $\beta \in \mathbb{C}^N$ with $\beta_i = 0$ for every $i \leq j$ and consider the map

$$\begin{aligned} f_\beta : \mathbb{C}^j &\to \mathbb{C} \\ \gamma &\mapsto \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle}. \end{aligned}$$

The application f_β is holomorphic and non constant. So there exists $\gamma_0 \in \mathbb{C}^j$ such that $|{\gamma_0}^\alpha \lambda^\beta e^{\langle \gamma_0, b\rangle}| = 1$. Since, $\mathbb{T} \setminus D$ is a dense set in $\mathbb{T}$, the vector γ_0 is an accumulation point of $\mathbb{T} \setminus D$. Thus, by [BGE06, Proposition 2.4], we get that the set

$$\Big\{ e_\gamma;\text{ with } \gamma \text{ such that } \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle} \in \mathbb{T} \setminus D \Big\}$$

spans a dense subspace in $H(\mathbb{C}^j)$. It is then easy to see that the set defined in (3.2.3) spans a dense subspace in $H(\mathbb{C}^N)$. In particular, we have shown that the set of eigenvectors of T associated to eigenvalues belonging to $\mathbb{T} \setminus D$ span a dense subspace in $H(\mathbb{C}^N)$. So, the hypothesis of Theorem 1.3.9 are satisfied and hence T is strongly mixing in the gaussian sense. $\square$

The following remark will be useful for the next proof and in the rest of the thesis.

Remark 3.2.8. Recall the Cauchy's formula for holomorphic functions in $\mathbb{C}^N$,

$$D^\alpha f(z_1, \dots, z_N) = \frac{\alpha!}{(2\pi i)^N} \int_{|w_1 - z_1| = r_1} \cdots \int_{|w_N - z_N| = r_N} \frac{f(w_1, \dots, w_N)}{\prod_{i=1}^N (w_i - z_i)^{\alpha_i + 1}}\, dw_1 \dots dw_N.$$

Therefore, we can estimate the supremum of $D^\alpha f$ over a set of the form $B(z_1, r_1) \times \cdots \times B(z_N, r_N)$, where $B(z_j, r_j)$ denotes the closed disk of center $z_j \in \mathbb{C}$ and radius r_j. Fix positive real numbers $\varepsilon_1, \dots, \varepsilon_N$, then

$$\|D^\alpha f\|_{\infty, B(z_1, r_1) \times \cdots \times B(z_N, r_N)} \leq \frac{\alpha!}{(2\pi)^N} \frac{\|f\|_{\infty, B(z_1, r_1 + \varepsilon_1) \times \cdots \times B(z_N, r_N + \varepsilon_N)}}{\varepsilon_1^{\alpha_1 + 1} \dots \varepsilon_N^{\alpha_N + 1}}. \tag{3.2.4}$$

Proof. (of Theorem 3.2.4) Part a) is proved by Lemmas 3.2.6 and 3.2.7.

b) Suppose that $b_l \neq 0$ for some l such that $\lambda_l = 1$. We will prove that T is a mixing operator, i.e., that for every pair U and V of non empty open sets for the local uniform topology of $H(\mathbb{C}^N)$, there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$. Let f and g be two holomorphic functions on $H(\mathbb{C}^N)$, L be a compact set of $\mathbb{C}^N$ and θ a positive real number. We can assume that

$$U = \{h \in H(\mathbb{C}^N) \,:\, \|f - h\|_{\infty, L} < \theta\} \text{ and } V = \{h \in H(\mathbb{C}^N) \,:\, \|g - h\|_{\infty, L} < \theta\},$$

and that g is a polynomial and that L is a closed ball of $(\mathbb{C}^N, \|\cdot\|_\infty)$. We do so because we can define a right inverse map over the set of polynomials. Since $T = C_\phi \circ D^\alpha$, we can define

$$I^\alpha(z^\beta) = \frac{\beta!}{(\alpha + \beta)!} z^{\alpha + \beta}.$$

Thus, $S = I^\alpha \circ C_{\phi^{-1}}$ is a right inverse for T when restricted to polynomials. Hence, we assume that $L = B(0,r) \times B(0,r) \times \cdots \times B(0,r)$, for some $r > 0$ and denote $\phi_i(z) = \lambda_i z + b_i$, for $z \in \mathbb{C}$. We get that $\phi(z_1, \ldots, z_N) = (\phi_1(z_1) \ldots, \phi_N(z_N))$ and $\phi_i(B(z_i, r_i)) = B(\phi_i(z_i), |\lambda_i| r_i)$.

Now, suppose that P is a polynomial in $\mathbb{C}^N$. Applying the inequality (3.2.4) several times, in which each time we use it we divide each ε_i by 2, we get that

$$\begin{aligned}
\|g - T^n P\|_{\infty,L} &= \left\|C_\phi \circ D^\alpha(Sg - T^{n-1}P)\right\|_{\infty,L} = \left\|D^\alpha(Sg - T^{n-1}P)\right\|_{\infty,\phi(L)} \\
&= \left\|D^\alpha(Sg - T^{n-1}P)\right\|_{\infty,\prod B(b_i,|\lambda_i|r)} \\
&\leq \frac{\alpha!}{(2\pi)^N \varepsilon_1^{\alpha_1+1} \ldots \varepsilon_N^{\alpha_N+1}} \left\|Sg - T^{n-1}P\right\|_{\infty,\prod B(b_i,|\lambda_i|r+\varepsilon_i)} \\
&\leq \frac{\alpha!}{(2\pi)^N \varepsilon_1^{\alpha_1+1} \ldots \varepsilon_N^{\alpha_N+1}} \left\|C_\phi \circ D^\alpha(S^2 g - T^{n-2}P)\right\|_{\infty,\prod B(b_i,|\lambda_i|r+\varepsilon_i)} \\
&\leq \frac{\alpha!}{(2\pi)^N \varepsilon_1^{\alpha_1+1} \ldots \varepsilon_N^{\alpha_N+1}} \left\|D^\alpha(S^2 g - T^{n-2}P)\right\|_{\infty,\prod B((\lambda_i+1)b_i,|\lambda_i|(|\lambda_i|r+\varepsilon_i))} \\
&\leq \frac{2^{|\alpha|+N}\alpha!^2}{(2\pi)^{2N} \varepsilon_1^{2(\alpha_1+1)} \ldots \varepsilon_N^{2(\alpha_N+1)}} \left\|S^2 g - T^{n-2}P\right\|_{\infty,\prod B((\lambda_i+1)b_i,|\lambda_i|(|\lambda_i|r+\varepsilon_i)+\frac{\varepsilon_i}{2})}.
\end{aligned}$$

Thus following, we get that

$$\|g - T^n P\|_{\infty,L} \leq \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN} \varepsilon_1^{n(\alpha_1+1)} \ldots \varepsilon_N^{n(\alpha_N+1)}} \|S^n g - P\|_{\infty,\prod B\left(\phi_i^n(0),|\lambda_i|^n r+\varepsilon_i \sum_{k=0}^{n-1} \frac{|\lambda_i|^k}{2^{n-k-1}}\right)}.$$

Let us denote by l, the coordinate of ϕ that is a translation in $\mathbb{C}$. Thus, we have that $\lambda_l = 1$ and $b_l \neq 0$. This implies that

$$B\left(\phi_l^n(0), |\lambda_l|^n r + \varepsilon_l \sum_{k=0}^{n-1} \frac{|\lambda_l|^k}{2^{n-k-1}}\right) = B\left(nb_l, r + \varepsilon_l \sum_{k=0}^{n-1} \frac{1}{2^k}\right) \subset B\left(nb_l, r + 2\varepsilon_l\right).$$

Fix $n_0 \in \mathbb{N}$, such that $B(0,r) \cap B\left(nb_l, r+2\varepsilon_l\right) = \emptyset$ for all $n \geq n_0$. Now, take $\delta_n > 0$ and Λ_n a ball of $(\mathbb{C}^N, \|\cdot\|_\infty)$, such that $[L + \delta_n] \cap [\Lambda_n + \delta_n] = \emptyset$ for all $n \geq n_0$ and

$$\prod_{i=1}^N B\left(\phi_l^n(0), |\lambda_l|^n r + \varepsilon_l \sum_{k=0}^{n-1} \frac{|\lambda_l|^k}{2^{n-k-1}}\right) \subset \Lambda_n.$$

Also, denote by

$$K_n = \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN} \varepsilon_1^{n(\alpha_1+1)} \ldots \varepsilon_N^{n(\alpha_N+1)}}.$$

Then, use Theorem 3.2.2 with $h_n = \chi_{L+\delta_n} f + \chi_{\Lambda_n+\delta_n} S^n g$. We get a polynomial P_n such that

$$\|f - P_n\|_L < \theta \text{ and } \|S^n g - P_n\|_{\Lambda_n} < \frac{\theta}{K_n}.$$

Hence,

$$\|f - P_n\|_L < \theta \text{ and } \|g - T^n P_n\|_L < \theta.$$

Thus, $P_n \in U \cap T^{-n}V$ for all $n \geq n_0$ and T is a mixing operator as we wanted to prove.

c) Let $\frac{b}{1-\lambda} = (\frac{b_1}{1-\lambda_1}, \dots, \frac{b_N}{1-\lambda_N})$ where, if $b_j = 0$ and $\lambda_j = 0$ for some $j = 1, \dots, N$, we will understand that $\frac{b_j}{1-\lambda_j} = 0$. Then $\frac{b}{1-\lambda}$ is a fixed point of ϕ, and thus

$$T^n f\left(\frac{b}{1-\lambda}\right) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\frac{b}{1-\lambda}\right).$$

Applying the Cauchy estimates we obtain

$$\left|T^n f\left(\frac{b}{1-\lambda}\right)\right| \leq |\lambda^\alpha|^{\frac{n(n-1)}{2}} \left|D^{n\alpha} f\left(\frac{b}{1-\lambda}\right)\right| \leq \frac{|\lambda^\alpha|^{\frac{n(n-1)}{2}}(n\alpha)!}{r^{n|\alpha|}} \sup_{\|z\|\leq r} |f(z)| \underset{n\to\infty}{\longrightarrow} 0.$$

Since the evaluation at the vector $\frac{b}{1-\lambda}$ is a continuous functional, this implies that the orbit of f under T is not dense. □

Notice that in case *b*) of Theorem 3.2.4 we do not know if the operator $C_\phi \circ D^\alpha$ is strongly mixing in the gaussian sense or even frequently hypercyclic. If $|\lambda_i| \leq 1$ for $1 \leq i \leq N$, we are able to show that the operator is frequently hypercyclic. To achieve this we prove that $C_\phi \circ D^\alpha$ is Runge transitive.

Definition 3.2.9. An operator T on a Fréchet space X is called Runge transitive if there is an increasing sequence (p_n) of seminorms defining the topology of X and numbers $N_m \in \mathbb{N}$, $C_{m,n} > 0$ for $m, n \in \mathbb{N}$ such that:

1. for all $m, n \in \mathbb{N}$ and $x \in X$,

$$p_m(T^n x) \leq C_{m,n} p_{n+N_m}(x)$$

2. for all $m, n \in \mathbb{N}$, $x, y \in X$ and $\varepsilon > 0$ there is some $z \in X$ such that

$$p_n(z - x) < \varepsilon \text{ and } p_m(T^{n+N_m} z - y) < \varepsilon.$$

The concept of Runge transitivity was introduced by Bonilla and Grosse-Erdmann. They proved in [BGE07, Theorem 3.3], that every Runge transitive operator on a Fréchet space is frequently hypercyclic. They also show that every translation operator on $H(\mathbb{C})$ is Runge transitive. However, the differentiation operator on $H(\mathbb{C})$ is not Runge transitive, even though we know that it is strongly mixing in the gaussian sense. Now, we prove that some of the operators which are included in the case *b*) are frequently hypercyclic.

Proposition 3.2.10. *Let T be the operator on $H(\mathbb{C}^N)$, defined by $Tf(z) = C_\phi \circ D^\alpha f(z)$, with $\alpha \neq 0$, $\phi(z) = (\lambda_1 z_1 + b_1, \dots, \lambda_N z_N + b_N)$ and $\lambda_i \neq 0$ for all i, $1 \leq i \leq N$. Then, if $|\lambda_i| \leq 1$ for every i, $1 \leq i \leq N$ and we have that $b_j \neq 0$ and $\lambda_j = 1$ for some j, $1 \leq j \leq N$, then T is Runge transitive.*

Proof. Define the increasing sequence of seminorms

$$p_m(f) = \sup_{\prod_{i=1}^N B(0, r_i(m))} |f(z)|,$$

where the radius $r_i(m)$ are defined as follows:

$$r_i(m) = \begin{cases} |b_i|m & \text{if } b_i \neq 0 \\ m & \text{if } b_i = 0 \end{cases}$$

We will prove that both conditions of the Definition 3.2.9 are satisfied with $N_m = m+1$. For the first condition, we proceed as in the proof of part c) of Theorem 3.2.4. We will apply several times the Cauchy inequalities (3.2.4) with ε_i defined as

$$\varepsilon_i = \begin{cases} \frac{|b_i|}{2} & \text{if } b_i \neq 0 \\ \frac{1}{2} & \text{if } b_i = 0 \end{cases}$$

and in each step we divide it by 2. So, we get that

$$p_m(T^n f) \leq \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN}\varepsilon_1^{n(\alpha_1+1)}\dots\varepsilon_N^{n(\alpha_N+1)}} \sup_{\Lambda} |f(z)|,$$

where $\Lambda = \prod B\left(\phi_i^n(0), |\lambda_i|^n r_i(m) + \varepsilon_i \sum_{k=0}^{n-1} \frac{|\lambda_i|^k}{2^{n-k-1}}\right)$.

Since $|\lambda_i| \leq 1$ for every i, $1 \leq i \leq N$, we obtain that

$$|\phi_i^n(0)| = \left| b_i \sum_{k=0}^{n-1} \lambda_i^k \right| \leq |b_i|n,$$

and that

$$|\lambda_i|^n r_i(m) + \varepsilon_i \sum_{k=0}^{n-1} \frac{|\lambda_i|^k}{2^{n-k-1}} \leq r_i(m) + 2\varepsilon_i.$$

From here it is easy to prove that $\Lambda \subseteq \prod B(0, r_i(n+m+1))$. Thus, if we denote

$$C_{m,n} = \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN}\varepsilon_1^{n(\alpha_1+1)}\dots\varepsilon_N^{n(\alpha_N+1)}},$$

we get that

$$p_m(T^n f) \leq C_{m,n} p_{n+m+1}(f).$$

Suppose that ε is a positive number, n and m are two integer numbers and that f, g are two holomorphic functions on $H(\mathbb{C}^N)$, we want to prove that there exists some function $h \in H(\mathbb{C}^N)$ such that

$$p_n(f-h) < \varepsilon \text{ and } p_m(T^{n+m+1}h - g) < \varepsilon.$$

Similarly, for the second condition we can estimate $p_m(T^{n+m+1}h - g)$ in the same way we did previously by making use of the right inverse for T. We get that

$$p_m(T^{n+m+1}h - g) \leq C \sup_{\Gamma} |S^{n+m+1}g - h|$$

where C is some positive constant and

$$\Gamma = \prod B\left(\phi_i^{n+m}(0), |\lambda_i|^{n+m+1} r_i(m) + \varepsilon_i \sum_{k=0}^{n+m} \frac{|\lambda_i|^k}{2^{n-k-1}}\right).$$

To assure the existence of such function h, by Runge's Theorem 3.2.2, it is enough to prove that $\Gamma \cap \prod B(0, r_i(n)) = \emptyset$. We study this sets in the j-th coordinate. We get that

$$\Gamma_j = B(b_j(n+m), r_j(m) + 2\varepsilon_j) = B(b_j(n+m), |b_j|(m+1)),$$

which is disjoint from $B(0, |b_j|n)$. Then, we have proved that the operator T is Runge transitive, hence it is frequently hypercyclic. $\square$

3.2.2 The non-diagonal case

We are now interested in the case in which the automorphism $\phi(z) = Az + b$, is given by any invertible matrix $A \in \mathbb{C}^{N\times N}$. Let $v \neq 0$ be any vector in $\mathbb{C}^N$ and let T be the operator on $H(\mathbb{C}^N)$ defined by

$$Tf(z) = C_\phi \circ D_v f(z) = D_v f(Az + b),$$

where $D_v f$ is the differential operator in the direction of v,

$$D_v f(z_0) = \lim_{s\to 0} \frac{f(z_0 + sv) - f(z_0)}{s} = \nabla f(z_0) \cdot v = df(\phi(z_0))(v).$$

The next two remarks show that we may consider a simplified version of the operator T.

Remark 3.2.11. We can assume that the matrix A is given in its Jordan form. Indeed, let Q be an invertible matrix in $\mathbb{C}^{N\times N}$ such that $A = QJQ^{-1}$, where J is the Jordan form of A. Also let $c = Q^{-1}b$ and denote $Q^*(f)(z) = f(Qz)$ for $f \in H(\mathbb{C}^N)$. Thus, we have that

$$Q^*(C_\phi \circ D_v f)(z) = \nabla f(AQz + b) \cdot v.$$

If we denote $\psi(z) = Jz + c$ and $w = Q^{-1}v$ then,

$$(C_\psi \circ D_w)Q^*(f)(z) = \nabla f(Q(Jz + c)) \cdot Qw = \nabla f(AQz + b) \cdot v.$$

We have proved that the following diagram commutes

$$\begin{tikzcd} H(\mathbb{C}^N) \arrow[r, "C_\phi \circ D_v"] \arrow[d, "Q^*"'] & H(\mathbb{C}^N) \arrow[d, "Q^*"] \\ H(\mathbb{C}^N) \arrow[r, "C_\psi \circ D_w"'] & H(\mathbb{C}^N) \end{tikzcd}$$

This shows that $C_\phi \circ D_v$ is linearly conjugate to $C_\psi \circ D_w$.

Remark 3.2.12. We can assume that $b = 0$ if the affine linear map ϕ has a fixed point $z_0 = \phi(z_0)$. Indeed, if we denote $\varphi(z) = Az$ then,

$$\tau_{z_0}(C_\phi \circ D_v)(f)(z) = D_v(f)(A(z + z_0) + b) = \tau_{z_0} D_v(f)(Az) = (C_\varphi \circ D_v)\tau_{z_0}(f)(z).$$

We have that the following diagram commutes

$$\begin{tikzcd} H(\mathbb{C}^N) \arrow[r, "C_\phi \circ D_v"] \arrow[d, "\tau_{z_0}"'] & H(\mathbb{C}^N) \arrow[d, "\tau_{z_0}"] \\ H(\mathbb{C}^N) \arrow[r, "C_\varphi \circ D_v"'] & H(\mathbb{C}^N) \end{tikzcd}$$

We conclude that $C_\phi \circ D_v$ is linearly conjugate to $C_\varphi \circ D_v$.

The first two results of this section deal with affine transformations that have fixed points.

Proposition 3.2.13. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let v be a nonzero vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ has a fixed point and that*

$$\lim_{k\to\infty} k! \prod_{i=0}^{k-1} \|A^i v\| < +\infty.$$

Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is not hypercyclic.

Consequently, $C_\phi \circ D_v$ is not hypercyclic if v belongs to an invariant subspace M of A such that the spectral radius of the restriction, $r(A|_M)$, is less than 1. This happens in particular if $r(A) < 1$ or if v is an eigenvector of A associated to an eigenvalue of modulus strictly less than 1.

Proof. We denote by $d^k f(z)$ to the k-th differential of a function f at z, which is a k-homogenous polynomial, and we denote by $\left(d^k f\right)^\vee(z)$ to the associated symmetric k-linear form.

It is not difficult to see that the orbits of the operator $C_\phi \circ D_v$ are determined by

$$(C_\phi \circ D_v)^k f(z) = \left(d^k f\right)^\vee(\phi^k z)(v, Av, \ldots, A^{k-1}v).$$

Assume that z_0 is a fixed point of ϕ, then applying the Cauchy's inequalities we get

$$\begin{aligned}|(C_\phi \circ D_v)^k f(z_0)| &= |\left(d^k f\right)^\vee(\phi^k z_0)(v, Av, \ldots, A^{k-1}v)| = |\left(d^k f\right)^\vee(z_0)(v, Av, \ldots, A^{k-1}v)| \\ &\le k! \prod_{i=0}^{k-1} \|A^i v\| \sup_{|z-z_0|<1} |f(z)|.\end{aligned}$$

Therefore $\{(C_\phi \circ D_v)^k f(z_0)\}$ is a bounded set of $\mathbb{C}$. Since the evaluation at z_0 is continuous, $C_\phi \circ D_v$ cannot have dense orbits.

For the last assertion, first note that if $J = Q^{-1}AQ$ is the Jordan form of A, we have that $w = Q^{-1}v$ belongs to the invariant subspace $Q^{-1}M$ of J and that $r := r(J|_{Q^{-1}M}) < 1$. By Remarks 3.2.11 and 3.2.12 it suffices to prove that $C_J \circ D_w$ is not hypercyclic.

It is not difficult to show that for every $i \ge N$,

$$\|J^i w\| \le c r^{i-N} i^N \|w\|,$$

where c is a constant that depends only on r and N. Therefore,

$$\begin{aligned}k! \prod_{i=0}^{k-1} \|J^i w\| &\le k! \prod_{i=0}^{N-1} \|J^i w\| \prod_{i=N}^{k-1} c r^{i-N} i^N \|w\| \\ &\le (k!)^{N+1} \|J\|^{(N+1)N/2} c^{k-N} \|w\|^k r^{(k-N)(k-N-1)/2} \to 0,\end{aligned}$$

which implies that $C_J \circ D_w$ is not hypercyclic by the first part of the proposition. □

In opposition to the previous result, if the matrix A is expansive when restricted to an invariant subspace then the operator is strongly mixing in the gaussian sense. This assumption is similar to the hypothesis of the results in the previous sections. Indeed, in the one dimensional case we have that $\phi(z) = \lambda z + b$ and if $|\lambda| \ge 1$, then the operator $C_\phi \circ D$ is strongly mixing in the gaussian sense. Here, the linear part of the composition operator is expansive. This situation still holds in the diagonal case in $H(\mathbb{C}^N)$. In this last case, we have that $\phi(z_1, \ldots, z_N) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$. Suppose that α is a multi-index of modulus one, i.e. that D^α is a partial derivative, then the hypothesis $|\lambda^\alpha| \geq 1$ turns out to be exactly the same as imposing that the linear part of ϕ is expansive on the subspace spanned by α. The proper result reads as follows.

Proposition 3.2.14. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let $v \neq 0$ be a vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ has a fixed point and that v belongs to a subspace M that reduces A and such that $\|(A|_M)^{-1}\| < 1$. Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is strongly mixing in the gaussian sense.*

Proof. We will show that the hypothesis of the Theorem 1.3.10 are fulfilled, taking as dense sets the polynomials in N complex variables. It is clear that $\sum_n T^n f$ converges unconditionally for every polynomial f. Now we will define a right inverse for $C_\phi \circ D_v$, but first we set some notation. Let us denote the fixed point of ϕ by z_0. Let us denote by π_1 to the orthogonal projection over M, $\pi_2 = I - \pi_1$ the orthogonal projection over $M^\perp$. Set $\mu(z) = \frac{\langle z,v\rangle}{\|v\|^2}$. We have that $z \mapsto \mu(z)v$ is the orthogonal projection over $span\{v\}$, and we denote $\tilde{\pi} = \pi_1 - \mu(z)v$. Finally, set $\phi_i(z) = Az + \pi_i(b)$, for $i = 1, 2$. Since, M reduces A, we have that ϕ_i is invertible and that $\pi_i(z_0)$ is a fixed point of ϕ_i, for $i = 1, 2$.

We define now for each $g \in H(\mathbb{C}^N)$,

$$Rg(z) = \int_{\mu(z_0)}^{\mu(z)} g(\phi_1^{-1}(tv + \tilde{\pi}(z)) + \pi_2(z))dt,$$

and $C(g)(z) = g(\pi_1(z) + \phi_2^{-1}(\pi_2(z)))$. Note that $R \circ C = C \circ R$. Finally, let $S = C \circ R$. Observe that,

$$Sg(z) = \int_{\mu(z_0)}^{\mu(z)} g(\phi^{-1}(tv + \tilde{\pi}(z) + \pi_2(z)))dt.$$

We have that

$$\begin{aligned}
D_v Sg(z) &= \lim_{s\to 0} \frac{Sg(z+sv) - Sg(z)}{s} \\
&= \lim_{s\to 0} \frac{1}{s}\left[\int_{\mu(z_0)}^{\mu(z+sv)} g(\phi^{-1}(tv + \tilde{\pi}(z) + \pi_2(z)))dt - \int_{\mu(z_0)}^{\mu(z)} g(\phi^{-1}(tv + \tilde{\pi}(z) + \pi_2(z)))dt\right] \\
&= \lim_{s\to 0} \frac{1}{s}\int_{\mu(z)}^{\mu(z)+s} g(\phi^{-1}(tv + \tilde{\pi}(z) + \pi_2(z)))dt \\
&= g(\phi^{-1}(\mu(z)v + \tilde{\pi}(z) + \pi_2(z))) \\
&= g(\phi^{-1}z).
\end{aligned}$$

Thus, $[C_\phi \circ D_v] \circ Sg = g$ for every $g \in H(\mathbb{C}^N)$. To conclude the proof we need to show that $\sum_n S^n g$ converges unconditionally for every polynomial g.

First we will bound the supremum of $|Rg|$ on $B(\pi_1 z_0, r) \times B(\pi_2 z_0, s)$, for a fixed polynomial g. Suppose that $z \in B(\pi_1 z_0, r) \times B(\pi_2 z_0, s)$ and that $t \in [\mu(z_0), \mu(z)]$ i.e. t lives in the complex segment from $\mu(z_0)$ to $\mu(z)$. Then we have that

$$\begin{aligned}\|tv+\tilde{\pi}(z)-\pi_1 z_0\|^2 &= \|(t-\mu(z_0))v+\tilde{\pi}(z-z_0)\|^2\\ &= |t-\mu(z_0)|^2\|v\|^2+\|\tilde{\pi}(z-z_0)\|^2\\ &\le |\mu(z)-\mu(z_0)|^2\|v\|^2+\|\tilde{\pi}(z-z_0)\|^2+\\ &= \|\pi_1(z-z_0)\|^2<r^2.\end{aligned}$$

Also, suppose that $\sigma:=\|(A|_M)^{-1}\|<1$. We get that

$$\begin{aligned}\|\phi_1^{-1}(\pi_1(z))-\pi_1(z_0)\| &= \|\phi_1^{-1}(\pi_1(z))-\phi_1^{-1}(\pi_1(z_0))\|\\ &= \|A^{-1}(\pi_1(z)-\pi_1(b))-A^{-1}(\pi_1(z_0)-\pi_1(b))\|\\ &\le \left\|(A|_M)^{-1}\right\|\,\|\pi_1(z)-\pi_1(z_0)\| = \sigma r.\end{aligned}$$

Gathering the previous statements we get that

$$\begin{aligned}|Rg(z)| &\le |\mu(z)-\mu(z_0)| \sup_{t\in[\mu(z_0),\mu(z)]} |g(\phi_1^{-1}(tv+\tilde{\pi}(z))+\pi_2(z))|\\ &\le \frac{r}{\|v\|^2}\sup_{w\in B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)} |g(\phi_1^{-1}(\pi_1(w))+\pi_2(w))| \le \frac{r}{\|v\|^2}\sup_{w\in B(\pi_1 z_0,\sigma r)\times B(\pi_2 z_0,s)}|g(w)|.\end{aligned}$$

Thus, we have proved that

$$\sup_{B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)}|Rg| \le \frac{r}{\|v\|^2}\sup_{B(\pi_1 z_0,\sigma r)\times B(\pi_2 z_0,s)}|g|.$$

Following by induction we obtain that

$$\begin{aligned}\sup_{B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)}|R^n g| &\le \frac{r}{\|v\|^2}\sup_{B(\pi_1 z_0,\sigma r)\times B(\pi_2 z_0,s)}|R^{n-1}g|\\ &\le \frac{r^n}{\|v\|^{2n}}\sigma^{\frac{n(n-1)}{2}}\sup_{B(\pi_1 z_0,\sigma^n r)\times B(\pi_2 z_0,s)}|g|.\end{aligned}$$

Finally, to conclude the proof we compute $\sup_{B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)}|S^n g(z)|$:

$$\begin{aligned}\sup_{B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)}|S^n g(z)| &= \sup_{B(\pi_1 z_0,r)\times B(\pi_2 z_0,s)}|R^n C^n g(z)|\\ &\le \frac{r^n}{\|v\|^{2n}}\sigma^{\frac{n(n-1)}{2}}\sup_{B(\pi_1 z_0,\sigma^n r)\times B(\pi_2 z_0,s)}|C^n g(z)|\\ &\le \frac{r^n}{\|v\|^{2n}}\sigma^{\frac{n(n-1)}{2}}\sup_{B(\pi_1 z_0,\sigma^n r)\times\phi_2^{-n}(B(\pi_2 z_0,s))}|g(z)|\end{aligned}$$

Since $\sigma<1$, we have proved that $\sum_n S^n g$ converges unconditionally for every polynomial g. Hence the operator $C_\phi\circ D_v$ is strongly mixing in the gaussian sense, as we wanted to prove. □

We turn now our discussion to the cases in which the affine linear map $\phi(z)=Az+b$ does not have a fixed point. This is equivalent to say that $b\notin Ran(I-A)$. Thus, 1 belongs to the spectrum of A. Then the Jordan form of A, which we denote by J, has a sub-block with ones in the principal diagonal and the first sub-diagonal and zeros elsewhere. It is easy to see that there exists some $k \in \mathbb{N}$, $k \leq N$ such that the canonical vector e_k does not belong to $Ran(I-J)$ and such that $b_k \neq 0$. This argument will be the key to show that ϕ is a runaway map, hence the operator $C_\phi \circ D_v$ is topologically transitive. The proof of this result is in the spirit of part (b) of Theorem 3.2.4.

Proposition 3.2.15. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let $v \neq 0$ be a vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ does not have a fixed point. Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is mixing.*

Proof. Due to the previous observations it is enough to prove that $C_\psi \circ D_w$ is topologically transitive if $\psi(z) = Jz + b$ with $b \notin Ran(I-J)$ and $w \in \mathbb{C}^N$, $w \neq 0$. We will denote $T = C_\psi \circ D_w$.

Given K_U, K_V two compact sets of $\mathbb{C}^N$, h_U, h_V two holomorphic functions in $H(\mathbb{C}^N)$ and θ a positive real number, we want to prove that there exists $k \in \mathbb{N}$ and $g \in H(\mathbb{C}^N)$ such that

$$\|g - h_U\|_{K_U} < \theta \text{ and } \|(C_\psi \circ D_w)^k g - h_V\|_{K_V} < \theta. \tag{3.2.5}$$

We will use Runge's theorem to show the existence of such function g. As before, we denote by S the right inverse of D_w. We have that

$$\begin{aligned}
\sup_{K_V} |C_\psi \circ D_w g(z) - h_V(z)| &= \sup_{K_V} \left| C_\psi \left(D_w g(z) - C_{\psi^{-1}} h_V(z) \right) \right| \\
&= \sup_{C_\psi(K_V)} \left| D_w g(z) - C_{\psi^{-1}} h_V(z)) \right| \\
&= \sup_{J(K_V)+b} \left| D_w \left(g(z) - S \circ C_{\psi^{-1}} h_V(z) \right) \right| \\
&\leq \frac{\|w\| N}{\varepsilon_1^N} \sup_{J(K_V)+B_{\varepsilon_1}(b)} \left| g(z) - S \circ C_{\psi^{-1}} h_V(z) \right|.
\end{aligned}$$

Following in this way inductively, we will get an estimate of $\|(C_\psi \circ D_w)^k g - h_V\|_{K_V}$,

$$\sup_{K_V} \left| (C_\psi \circ D_w)^l g(z) - h_V(z) \right| \leq \alpha(l) \sup_{A_l} \left| g(z) - (S \circ C_{\psi^{-1}})^l h_V(z) \right|,$$

with $\alpha(l) > 0$ and $A_l = J^l(K_V) + \sum_{i=1}^l J^i(B(0,\varepsilon_i)) + \sum_{i=1}^l J^i(b)$.

It is enough to find some $l \in \mathbb{N}$ such that $K_U \cap A_l = \emptyset$. Without loss of generality we can assume that $e_1 \notin Ran(J-I)$ and $b_1 \neq 0$ (see the comments before the proposition). This means that J acts like the identity in the first coordinate.

Suppose that $K_V \subset \prod_{i=1}^N B(0,r_i)$, then if we project in the first coordinate and choose proper $\varepsilon_i > 0$ we obtain

$$\begin{aligned}
[A_l]_1 &= [J^l(K_V)]_1 + \sum_{i=1}^l [J^i(B(0,\varepsilon_i))]_1 + \sum_{i=1}^l [J^i(b)]_1 \\
&\subset B(0,r_1) + B(0, \sum_{i=1}^l \varepsilon_i) + lb_1 \\
&\subset B(0,R) + lb_1.
\end{aligned}$$

Thus, we will able to find $l_0 \in \mathbb{N}$ such that $[K_U]_1 \cap [A_l]_1 = \emptyset$ for all $l \geq l_0$. Therefore, by Runge's Theorem, there exists some $g_l \in H(\mathbb{C}^N)$ such that (3.2.5) is satisfied for all $l \geq l_0$. We have proved that the operator $C_\psi \circ D_w$ is mixing, as we wanted to prove.

$\square$

Chapter 4

Holomorphic functions on Banach spaces

4.1 Holomorphic functions of $\mathfrak{A}$-bounded type

In this chapter we recall the basic properties of holomorphic functions on Banach spaces, the best general reference here is [Din99]. We also introduce the spaces of entire functions $\mathcal{H}_{b\mathfrak{A}}(E)$ and convolution operators therein. We will work with hypercyclic operators on these spaces in the next chapters.

From now on E will be a complex Banach space. A mapping $P : E \to \mathbb{C}$ is a continuous k-homogeneous polynomial if there exists a (necessarily unique) continuous and symmetric k-linear form $L : E^k \to \mathbb{C}$ such that $P(z) = L(z, \dots, z)$ for all $z \in E$. For example, given $\gamma \in E'$, the function $P(z) = \gamma(z)^k$ is a $k-$homogeneous polynomial. The space of all continuous k-homogeneous polynomials from E to $\mathbb{C}$, endowed with the norm $\|P\|_{\mathcal{P}(^kE)} = \sup_{\|z\|_E=1} |P(z)|$ is a Banach space and it will be denoted by $\mathcal{P}(^kE)$. The space $\mathcal{P}(^0E)$ is just $\mathbb{C}$. The space of *finite type* polynomials, denoted by $\mathcal{P}_f(^kE)$, is the subspace of $\mathcal{P}(^kE)$ spanned by $\{\gamma(\cdot)^k\}_{\gamma\in E'}$.

A function $f : E \to \mathbb{C}$ is holomorphic if there exist k-homogeneous polynomials $d^kf(a)$ such that $f(x) = \sum_{k\geq 0} \frac{d^kf(a)}{k!}(x-a)$, where the series converges uniformly in some neighborhood around the point of expansion. We say that this series is the Taylor series of f around the point a. The space of holomorphic functions from E to $\mathbb{C}$ is denoted by $\mathcal{H}(E)$. The space of holomorphic functions whose Taylor series have infinite radius of uniform convergence is denoted $\mathcal{H}_b(E)$. Such functions are bounded on bounded sets, and are said to be of bounded type. The space $\mathcal{H}_b(E)$ is a Fréchet space when considered with the topology of uniform convergence on bounded sets of E.

Given $P \in \mathcal{P}(^kE)$, $a \in E$ and $0 \leq j \leq k$, let $P_{a^j} \in \mathcal{P}^{k-j}(E)$ be the polynomial defined by

$$P_{a^j}(x) = \overset{\vee}{P}(a^j, x^{k-j}) = \overset{\vee}{P}(\underbrace{a, ..., a}_{j}, \underbrace{x, ..., x}_{k-j}),$$

where $\overset{\vee}{P}$ is the unique symmetric k-linear form associated to P. The following equation describes the relation between the map P_{a^j} and the polynomials of the Taylor series of P around a,

$$\frac{d^{k-j}P}{(k-j)!}(a) = \binom{k}{j} P_{a^j}.$$

We write P_a instead of P_{a^1}. Let us recall the definition of a polynomial ideal [Flo01, Flo02].

Definition 4.1.1. A Banach ideal of scalar-valued continuous k-homogeneous polynomials, $k \geq 0$, is a pair $(\mathfrak{A}_k, \|\cdot\|_{\mathfrak{A}_k})$ such that:

(i) For every Banach space E, $\mathfrak{A}_k(E) = \mathfrak{A}_k \cap \mathcal{P}(^kE)$ is a linear subspace of $\mathcal{P}(^kE)$ and $\|\cdot\|_{\mathfrak{A}_k(E)}$ is a norm on it. Moreover, $(\mathfrak{A}_k(E), \|\cdot\|_{\mathfrak{A}_k(E)})$ is a Banach space.

(ii) If $T \in \mathcal{L}(E_1, E)$ and $P \in \mathfrak{A}_k(E)$, then $P \circ T \in \mathfrak{A}_k(E_1)$ with

$$\|P \circ T\|_{\mathfrak{A}_k(E_1)} \leq \|P\|_{\mathfrak{A}_k(E)} \|T\|^k.$$

(iii) $z \mapsto z^k$ belongs to $\mathfrak{A}_k(\mathbb{C})$ and has norm 1.

The concept of holomorphy type was introduced by Nachbin [Nac69]. We will use it in the following slightly modified version (see [Mur12]).

Definition 4.1.2. Consider the sequence $\mathfrak{A} = \{\mathfrak{A}_k\}_{k=0}^{\infty}$, where for each k, $\mathfrak{A}_k$ is a Banach ideal of k-homogeneous polynomials. We say that $\{\mathfrak{A}_k\}_k$ is a holomorphy type if there exist constants c, $c_{k,l}$ such that $c_{k,l} \leq c^k$ for every $0 \leq l \leq k$ and such that for every Banach space E, $P \in \mathfrak{A}_k(E)$ and $a \in E$,

$$P_{a^l} \text{ belongs to } \mathfrak{A}_{k-l}(E) \text{ and } \|P_{a^l}\|_{\mathfrak{A}_{k-l}(E)} \leq c_{k,l} \|P\|_{\mathfrak{A}_k(E)} \|a\|^l. \tag{4.1.1}$$

Remark 4.1.3. Sometimes we will require that the constants satisfy, for every k, l,

$$c_{k,l} \leq \frac{(k+l)^{k+l}}{(k+l)!} \frac{k!}{k^k} \frac{l!}{l^l}. \tag{4.1.2}$$

These constants are more restrictive than Nachbin's constants (the constants considered by Nachbin were of the form $c_{k,l} = \binom{k}{l} C^k$ for some fixed constant C), but, the constants $c_{k,l}$ of every usual example of holomorphy type satisfy (4.1.2).

Remark 4.1.4. Stirling's Formula states that $e^{-1} n^{n+1/2} \leq e^{n-1} n! \leq n^{n+1/2}$ for every $n \geq 1$, so given $\varepsilon > 0$, there exists a positive constant c_ε, such that

$$c_{k,l} \leq e^2 \Big(\frac{kl}{k+l}\Big)^{1/2} \leq c_\varepsilon (1+\varepsilon)^k,$$

for every $0 \leq l \leq k$.

There is a natural way to associate to a holomorphy type $\mathfrak{A}$ a class of entire functions of bounded type on a Banach space E, as the set of entire functions with infinite $\mathfrak{A}$-radius of convergence at zero, and hence at every point (see [CDM07, FJ09]).

Definition 4.1.5. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E be a Banach space. The space of entire functions of $\mathfrak{A}$-bounded type on E, $\mathcal{H}_{b\mathfrak{A}}(E)$ is the set of all entire functions $f \in \mathcal{H}(E)$ such that $d^k f(0) \in \mathfrak{A}_k(E)$ for every k and $\lim_{k\to\infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k}^{1/k} = 0$.

We consider in $\mathcal{H}_{b\mathfrak{A}}(E)$ the family of seminorms $\{p_s\}_{s>0}$, given by

$$p_s(f) = \sum_{k=0}^{\infty} s^k \left\| \frac{d^k f(0)}{k!} \right\|_{\mathfrak{A}_k},$$

for all $f \in \mathcal{H}_{b\mathfrak{A}}(E)$. It is easy to check that $(\mathcal{H}_{b\mathfrak{A}}(E), \{p_s\}_{s>0})$ is a Fréchet space.

Example 4.1.6. This example collects some of the spaces of entire functions of bounded type that may be constructed in this way. See the references given in each case for the definition and details.

(i) If we let $\mathfrak{A}_k = \mathcal{P}^k$, the ideal of all k-homogeneous continuous polynomials, then the topology induced on $\mathcal{H}_{b\mathfrak{A}}(E)$ by $\{p_s\}_{s>0}$ is equivalent to the usual topology of uniform convergence on bounded sets. Therefore $\mathcal{H}_{b\mathfrak{A}}(E) = \mathcal{H}_b(E)$.

(ii) If $\mathfrak{A}$ is the sequence of ideals of nuclear polynomials then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space of holomorphic functions of nuclear bounded type $H_{Nb}(E)$ defined by Gupta and Nachbin (see [Gup70]).

(iii) If E is a Hilbert space and $\mathfrak{A}$ is the sequence of ideals of Hilbert-Schmidt polynomials, then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space $H_{hs}(E)$ of entire functions of Hilbert-Schmidt type (see [Dwy71, Pet01]).

(iv) If $\mathfrak{A}$ is the sequence of ideals of approximable polynomials, then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space $H_{bc}(E)$ of entire functions of compact bounded type (see for example [Aro79, AB99]).

(v) If $\mathfrak{A}$ is the sequence of ideals of weakly continuous on bounded sets polynomials, then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space $H_{wu}(E)$ of weakly uniformly continuous holomorphic functions on bounded sets defined by Aron in [Aro79].

(vi) If $\mathfrak{A}$ is the sequence of ideals of extendible polynomials, then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space of extendible functions of bounded type defined in [Car01].

(vii) If $\mathfrak{A}$ is the sequence of ideals of integral polynomials, then $\mathcal{H}_{b\mathfrak{A}}(E)$ is the space of integral holomorphic functions of bounded type $H_{bI}(E)$ defined in [DGMZ04].

Finite type polynomials are dense in $\mathfrak{A}_k(E)$ in many cases. For example, finite type polynomials are dense in the spaces of nuclear, Hilbert-Schmidt and approximable polynomials. They are also dense in $\mathcal{P}(^kE)$ if E is c_0 or the Tsirelson space and in the spaces of integral and extendible polynomials if E is Asplund [CG11]. On the other hand, separability is a necessary condition to deal with hypercyclicity issues on $\mathcal{H}_{b\mathfrak{A}}(E)$ and, up to our knowledge, on every example of separable space of polynomials, finite type polynomials are dense.

We also note that a holomorphy type such that finite type polynomials are dense is essentially what is called an α-β-holomorphy type in [Din71] and a π_1-holomorphy type in [FJ09, BBFJ13].

Chapter 5

Strongly mixing convolution operators on Fréchet spaces of holomorphic functions

A theorem of Godefroy and Shapiro states that non-trivial convolution operators on the space of entire functions on $\mathbb{C}^n$ are hypercyclic. Moreover, it was shown by Bonilla and Grosse-Erdmann that they have frequently hypercyclic functions of exponential growth. On the other hand, in the infinite dimensional setting, the Godefroy-Shapiro theorem has been extended to several spaces of entire functions defined on Banach spaces. We prove that on all these spaces, non-trivial convolution operators are strongly mixing with respect to a gaussian probability measure of full support. For the proof we combine the results previously mentioned and we use techniques recently developed by Bayart and Matheron. We also obtain the existence of frequently hypercyclic entire functions of exponential growth.

Several criteria to determine if an operator is hypercyclic have been studied. It is known that a large supply of eigenvectors implies hypercyclicity. In particular, if the eigenvectors associated to eigenvalues of modulus less than 1 and the eigenvectors associated to eigenvalues of modulus greater than 1 span dense subspaces, then the operator is hypercyclic. This result is due to Godefroy and Shapiro [GS91]. They also prove there that non-trivial convolution operators, i.e. operators that commute with translations and which are not multiples of the identity, on the space of entire functions on $\mathbb{C}^n$ are hypercyclic. This result has also been extended to some spaces of entire functions on infinite dimensional Banach spaces (see [AB99, BBFJ13, CDM07, Pet01, Pet06]). The Godefroy - Shapiro theorem has been improved by Bonilla and Grosse-Erdmann. They showed that non-trivial convolution operators are even frequently hypercyclic, and have frequently hypercyclic entire functions satisfying some exponential growth condition (see [BGE06]).

Recent work developed by Bayart and Matheron [BM14] provides some other eigenvector criteria to determine whether a given continuous map $T : X \to X$ acting on a topological space X admits an ergodic probability measure, or a strong mixing one. When the measure is strictly positive on any non void open set of X, ergodic properties on T imply topological counterparts. In particular, if a continuous map $T : X \to X$ happens to be ergodic with respect to some Borel probability measure μ with full support, then almost every $x \in X$ (relative to μ) has a dense

T-orbit. Moreover, from Birkhoff's ergodic theorem, we can obtain frequent hypercyclicity.

In this chapter we study convolution operators on spaces of entire functions defined on Banach spaces. We show that under suitable conditions, non-trivial convolution operators are strongly mixing, and in particular, frequently hypercyclic. In the same spirit as Bonilla and Grosse-Erdmann, we also obtain the existence of frequently hypercyclic entire functions of exponential growth associated to these operators. We also prove the existence of frequently hypercyclic subspaces for a given non-trivial convolution operator, that is, the existence of closed infinite-dimensional subspaces in which every non-zero vector is a frequently hypercyclic function. Finally, we study particular cases of non-trivial convolution operators such as translations and partial differentiation operators. In this cases we obtain bounds of the exponential growth of the frequently hypercyclic entire functions.

5.1 Strongly mixing convolution operators

In this section we prove our first main theorem, which states that under some fairly general conditions on the space E and the holomorphy type $\mathfrak{A}$, non-trivial convolution operators on $\mathcal{H}_{b\mathfrak{A}}(E)$ are strongly mixing in the gaussian sense. First we recall the following definitions.

Given $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ a holomorphy type, the Borel transform is the operator $\beta : \mathcal{H}_{b\mathfrak{A}}(E)' \to \mathcal{H}(E')$ which assigns to each element $\varphi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ the holomorphic function $\beta(\varphi) \in \mathcal{H}(E')$, given by $\beta(\varphi)(\gamma) = \varphi(e^\gamma)$. The following proposition is well-known (see for example [Din71, p.264] or [FJ09, p.915]).

Proposition 5.1.1. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E be a Banach space such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Then the Borel transform is an injective linear transformation.*

We denote by $\tau_x(f) := f(x + \cdot)$ the translation operator by x, which is a continuous linear operator on $\mathcal{H}_{b\mathfrak{A}}(E)$ (see [CDM07, FJ09]). The following is the usual definition of convolution operator.

Definition 5.1.2. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E be a Banach space. A linear continuous operator T defined on $\mathcal{H}_{b\mathfrak{A}}(E)$ is a *convolution operator*, if for every $x \in E$ we have $T \circ \tau_x = \tau_x \circ T$. We say that T is trivial if it is a multiple of the identity.

The following proposition provides a description of convolution operators on $\mathcal{H}_{b\mathfrak{A}}(E)$. Its proof follows as [CDM07, Proposition 4.7].

Proposition 5.1.3. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E be a Banach space. Then for each convolution operator $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ there exists a linear functional $\varphi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ such that*

$$T(f) = \varphi * f,$$

for every $f \in \mathcal{H}_{b\mathfrak{A}}(E)$, where $\varphi * f(x) := \varphi(\tau_x f) = \varphi(f(x + \cdot))$.

Proof. Let $\varphi = \delta_0 \circ T$, i.e. $\varphi(f) = T(f)(0)$ for $f \in \mathcal{H}_{b\mathfrak{A}}(E)$. Then $\varphi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ and

$$T(f)(x) = [\tau_x T(f)](0) = T(\tau_x f)(0) = \varphi(\tau_x f) = \varphi * f(x),$$

for every $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ and $x \in E$. $\square$

The next lemma is the key to prove that convolution operators are strongly mixing and it we will be used throughout the thesis.

Lemma 5.1.4. *Let E be a Banach space with separable dual and let $\mathfrak{A}$ be a holomorphy type such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Let $\phi \in \mathcal{H}(E')$ not constant and $B \subset \mathbb{C}$. Suppose that there exist $\gamma_0 \in E'$ such that $\phi(\gamma_0)$ is an accumulation point of B. Then span$\{e^\gamma : \phi(\gamma) \in B\}$ is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$.*

Proof. Let $\Phi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ be a functional vanishing on $\{e^\gamma : \phi(\gamma) \in B\}$. Note that this means that $\beta(\Phi)$ vanishes on $\phi^{-1}(B)$. By Proposition 5.1.1, it suffices to show that $\beta(\Phi) = 0$.

Fix $\gamma_0 \in E'$ such that $\phi(\gamma_0)$ is an accumulation point of B. We claim that there exist a sequence of complex lines L_k, $k \in \mathbb{N}$, through γ_0 such that ϕ is not constant on each L_k and $\bigcup_k L_k$ is dense in E'. Indeed, let $\{U_k\}_{k\in\mathbb{N}}$, be open sets that form a basis of the topology of E'. Since ϕ is not constant, there exists, for each k, a complex line L_k through γ_0 that meets U_k and on which ϕ is not constant.

Now let $k \in \mathbb{N}$. Since ϕ is not constant on L_k, $\phi|_{L_k}$ is an open mapping, and hence γ_0 is an accumulation point of $\phi^{-1}(B) \cap L_k$. But, $\beta(\Phi)$ vanishes on $\phi^{-1}(B)$. Thus, $\beta(\Phi)$ also vanishes on L_k. Since $\bigcup_k L_k$ is dense in E', $\beta(\Phi) = 0$. □

We are now able to prove that convolution operators on $\mathcal{H}_{b\mathfrak{A}}(E)$ are strongly mixing in the gaussian sense.

Theorem 5.1.5. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E a Banach space with separable dual such that the finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. If $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is a non-trivial convolution operator, then T is strongly mixing in the gaussian sense.*

Proof. Let $\varphi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ be the linear functional defined in the proof of Proposition 5.1.3. Since T is not a multiple of the identity it follows that φ is not a multiple of δ_0. Also, the fact that φ is not a multiple of δ_0 implies that $\beta(\varphi)$ is not a constant function. Indeed, if $\beta(\varphi)$ were constant then $\lambda := \varphi(1) = \beta(\varphi)(0) = \beta(\varphi)(\gamma) = \varphi(e^\gamma)$ for all $\gamma \in E'$. But, on the other hand, $\lambda = \lambda\delta_0(e^\gamma)$ for all $\gamma \in E'$ and we would have that $\varphi = \lambda\delta_0$.

It is rather easy to find eigenvalues and eigenvectors for T. Given $\gamma \in E'$,

$$T(e^\gamma) = \varphi * e^\gamma = [x \mapsto \varphi(\tau_x e^\gamma)] = \varphi(e^\gamma)\left[x \mapsto e^{\gamma(x)}\right] = \beta(\varphi)(\gamma)e^\gamma.$$

By Theorem 1.3.9, it suffices to prove that the set of unimodular eigenvectors $\{e^\gamma \in \mathcal{H}_{b\mathfrak{A}}(E) : |\beta(\varphi)(\gamma)| = 1\}$ is big enough. Let us first prove that it is not empty. Define

$$V = \{\gamma \in E' : \ |\beta(\varphi)(\gamma)| < 1\} \ \text{ and } \ W = \{\gamma \in E' : |\beta(\varphi)(\gamma)| > 1\}.$$

Let us check that $V, W \subset E'$ are non void open sets. Indeed, if $V = \emptyset$, or $W = \emptyset$, then $\frac{1}{\beta(\varphi)}$, or $\beta(\varphi)$, would be a nonconstant bounded entire function. Since $\beta(\varphi)(E')$ is arcwise connected, we can deduce the existence of $\gamma_0 \in E'$ such that $|\beta(\varphi)(\gamma_0)| = 1$.

Let $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$. Then, $\beta(\varphi)(\gamma_0)$ is an accumulation point of $\mathbb{T} \setminus D$ and by Lemma 5.1.4 we get that the linear span of $\bigcup_{\lambda\in\mathbb{T}\setminus D} \ker(T - \lambda)$ is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$. By Theorem 1.3.9, it follows that T is strongly mixing in the gaussian sense, as we wanted to prove. □

5.2 Exponential growth conditions for frequently hypercyclic entire functions

In this section we show that for every convolution operator there exists a frequently hypercyclic entire function satisfying a certain exponential growth condition. First, we define and study a family of Fréchet subspaces of $\mathcal{H}_{b\mathfrak{A}}(E)$ consisting of functions of exponential type; and then we show that every convolution operator on $\mathcal{H}_{b\mathfrak{A}}(E)$ defines a frequently hypercyclic operator on these spaces.

Definition 5.2.1. A function $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ is said to be of M-exponential type if there exist some constant $C > 0$ such that $|f(x)| \leq Ce^{M\|x\|}$, for all $x \in E$. We say that f is of exponential type if it is of M-exponential type for some $M > 0$.

Now we define the subspaces of $\mathcal{H}_{b\mathfrak{A}}(E)$ consisting of functions of exponential type.

Definition 5.2.2. For $p > 0$, let us define the space

$$Exp^p_{\mathfrak{A}}(E) = \left\{ f \in \mathcal{H}_{b\mathfrak{A}}(E) : \limsup_{k\to\infty} \|d^k f(0)\|^{1/k}_{\mathfrak{A}_k} \leq p \right\},$$

endowed with the family of seminorms defined by

$$q_r(f) = \sum_{k=0}^{\infty} r^k \|d^k f(0)\|_{\mathfrak{A}_k} \quad \text{for } 0 < r < 1/p.$$

Below we collect some basic properties of the spaces $Exp^p_{\mathfrak{A}}(E)$. Their proof is standard.

Proposition 5.2.3. *Let p be a positive number and $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ a holomorphy type.*

(a) A function $f \in H(E)$ belongs to $Exp^p_{\mathfrak{A}}(E)$ if and only if $d^k f(0) \in \mathfrak{A}_k$ for all $k \in \mathbb{N}$ and $q_r(f) < \infty$, for all $0 < r < 1/p$.

(b) The space $(Exp^p_{\mathfrak{A}}(E), \{q_r\}_{r<1/p})$ is a Fréchet space that is continuously and densely embedded in $\mathcal{H}_{b\mathfrak{A}}(E)$.

(c) If E' is separable and finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k, then $Exp^p_{\mathfrak{A}}(E)$ is separable.

(d) Every function $f \in Exp^p_{\mathfrak{A}}(E)$ satisfies the following growth condition: for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x)| \leq C_\varepsilon e^{(p+\varepsilon)\|x\|}, \quad x \in E,$$

that is, f is of exponential type p.

In order to prove frequent hypercyclicity of convolution operators on $Exp^p_{\mathfrak{A}}(E)$, we need to introduce some structure on the sequence of polynomial ideals.

Definition 5.2.4. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ a holomorphy type and let E be a Banach space. We say that $\mathfrak{A}$ is weakly differentiable at E if there exist constants $c_{k,l} > 0$ such that, for $0 \leq l \leq k$, $P \in \mathfrak{A}_k(E)$ and $\varphi \in \mathfrak{A}_{k-l}(E)'$, the mapping $x \mapsto \varphi(P_{x^l})$ belongs to $\mathfrak{A}_l(E)$ and

$$\left\| x \mapsto \varphi(P_{x^l}) \right\|_{\mathfrak{A}_l(E)} \leq c_{k,l} \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

Remark 5.2.5. In the following, we will assume that $c_{k,l}$ satisfy 4.1.2. So given $\varepsilon > 0$, there exists a positive constant c_ε, such that

$$c_{k,l} \leq e^2 \Big(\frac{kl}{k+l} \Big)^{1/2} \leq c_\varepsilon (1+\varepsilon)^k,$$

for every $0 \leq l \leq k$.

Remark 5.2.6. Weak differentiability is a stonger condition than being a holomorphy type and was defined in [CDM12]. All the spaces of entire functions appearing in Example 4.1.6 are constructed with weakly differentiable holomorphy types satisfying (4.1.2), see [CDM12, Mur12]. The concept of weak differentiability is closely related to that of α-β-γ-holomorphy types in [Din71] and that of π_1-π_2-holomorphy types in [FJ09, BBFJ13].

Proposition 5.2.7. *Let p be a positive number, $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ a holomorphy type and let E be a Banach. Suppose that $\mathfrak{A}$ is weakly differentiable with constants $c_{k,l}$ satisfying (4.1.2). Then every convolution operator on $\mathcal{H}_{b\mathfrak{A}}(E)$, restricts to a convolution operator on $Exp^p_{\mathfrak{A}}(E)$.*

Proof. Let $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ be a convolution operator and $\varphi \in \mathcal{H}_{b\mathfrak{A}}(E)'$ such that $Tf = \varphi * f$. Suppose that $f = \sum_{k\in\mathbb{N}_0} P_k$ is in $Exp^p_{\mathfrak{A}}(E)$. We need to prove that for $r < 1/p$

$$q_r(\varphi * f) = \sum_{l=0}^{\infty} r^l \|d^l(\varphi * f)(0)\|_{\mathfrak{A}_l} < \infty.$$

Note that

$$\varphi * f(x) = \varphi(\tau_x f) = \varphi \left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} (P_k)_{x^l} \right) = \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \binom{k}{l} \varphi\left((P_k)_{x^l}\right).$$

This implies that

$$d^l(\varphi * f)(0)(x) = l! \sum_{k=l}^{\infty} \binom{k}{l} \varphi((P_k)_{x^l}).$$

Since φ is a continuous linear functional, there are positive constants c and M such that $\|\varphi\|_{\mathfrak{A}'_{k-l}} \leq cM^{k-l}$. Thus, given $\varepsilon > 0$ such that $r(1+\varepsilon) < 1/p$, by the above remark,

$$\begin{aligned}
q_r(\varphi * f) &= \sum_{l=0}^{\infty} r^l \|d^l(\varphi * f)(0)\|_{\mathfrak{A}_l} \\
&\leq \sum_{l=0}^{\infty} r^l l! \sum_{k=l}^{\infty} \binom{k}{l} \|x \mapsto \varphi((P_k)_{x^l})\|_{\mathfrak{A}_k} \\
&\leq \sum_{l=0}^{\infty} r^l l! \sum_{k=l}^{\infty} \binom{k}{l} c_{k,l} \|\varphi\|_{\mathfrak{A}'_{k-l}} \|P_k\|_{\mathfrak{A}_k} \\
&\leq c \sum_{k=0}^{\infty} \frac{\|d^k f(0)\|_{\mathfrak{A}_k}}{k!} \sum_{l=0}^{k} \binom{k}{l} c_{k,l}\, r^l\, l!\, M^{k-l} \\
&\leq c \sum_{k=0}^{\infty} \|d^k f(0)\|_{\mathfrak{A}_k} r^k c_\varepsilon (1+\varepsilon)^k \sum_{l=0}^{k} \frac{\left(\frac{M}{r}\right)^{k-l}}{(k-l)!} \\
&\leq c\, c_\varepsilon\, e^{(M/r)} \sum_{k=0}^{\infty} \|d^k f(0)\|_{\mathfrak{A}_k} (r(1+\varepsilon))^k \\
&= c\, c_\varepsilon\, e^{(M/r)}\, q_{r(1+\varepsilon)}(f) < \infty.
\end{aligned}$$

□

Remark 5.2.8. For $\gamma \in E'$, we have $d^k(e^\gamma)(0) = \gamma^k$, and then, since $\|\gamma^k\|_{\mathfrak{A}_k} = \|\gamma\|^k$,

$$\limsup_{k\to\infty} \|d^k e^\gamma(0)\|_{\mathfrak{A}_k}^{1/k} = \|\gamma\|_{E'}.$$

This implies that $e^\gamma \in Exp^p_{\mathfrak{A}}(E)$ if and only if $\|\gamma\| \leq p$. Thus, for $\varphi \in Exp^p_{\mathfrak{A}}(E)'$, we can define the Borel transform $\beta(\varphi)(\gamma) = \varphi(e^\gamma)$, for all $\gamma \in E'$ with $\|\gamma\| \leq p$. Moreover, the function $\beta(\varphi)$ is holomorphic on the set $p\,B_{E'}$.

The next proposition is the analogue of Proposition 5.1.1 for the Borel transform restricted to $Exp^p_{\mathfrak{A}}(E)$.

Proposition 5.2.9. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E a Banach space such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Then the Borel transform $\beta : Exp^p_{\mathfrak{A}}(E)' \to \mathcal{H}(pB_{E'})$ is an injective linear transformation.*

Now, we can restate Lemma 5.1.4, for the space $Exp^p_{\mathfrak{A}}(E)$. Its proof is similar.

Lemma 5.2.10. *Let p be a positive number, let E be a Banach space with separable dual and let $\mathfrak{A}$ be a holomorphy type such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Let $\phi \in \mathcal{H}(pB_{E'})$ not constant and $B \subset \mathbb{C}$. Suppose that there exist $\gamma_0 \in p\,B_{E'}$ such that $\phi(\gamma_0)$ is an accumulation point of B. Then span$\{e^\gamma : \|\gamma\| < p, \phi(\gamma) \in B\}$ is dense in $Exp^p_{\mathfrak{A}}(E)$.*

Now we are able to prove that for non-trivial convolution operators on $\mathcal{H}_{b\mathfrak{A}}(E)$ there exist frequently hypercyclic entire function satisfying certain exponential growth conditions.

Given a non-trivial convolution operator T defined on $\mathcal{H}_{b\mathfrak{A}}(E)$, let us define

$$\alpha_T = \inf\{\|\gamma\|,\, \gamma \in E' \text{ such that } |T(e^\gamma)(0)| = 1\}.$$

Theorem 5.2.11. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and let E be a Banach space with separable dual such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Suppose that $\mathfrak{A}$ is weakly differentiable with constants $c_{k,l}$ satisfying (4.1.2). Let $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ be a non-trivial convolution operator. Then, for any $\varepsilon > 0$, T admits a frequent hypercyclic function $f \in Exp_{\mathfrak{A}}^{\alpha_T+\varepsilon}(E)$.*

Proof. Fix $\gamma_0 \in E'$ such that $\alpha_T \leq \|\gamma_0\| < \alpha_T + \varepsilon$ and $|T(e^{\gamma_0})(0)| = 1$. Consider $p = \alpha_T + \varepsilon$. It is enough to prove that T is frequently hypercyclic on $Exp_{\mathfrak{A}}^{p}(E)$.

The Proposition 5.2.7 allows us to restrict the operator T to the space $Exp_{\mathfrak{A}}^{p}(E)$. Since e^{γ} is an eigenvector of T with eigenvalue $T(e^{\gamma})(0)$, it is enough to show, by Theorem 1.3.9, that for every Borel set $D \subset \mathbb{T}$, such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the linear span of $\{e^{\gamma} : \|\gamma\| < p,\ T(e^{\gamma})(0) \in \mathbb{T} \setminus D\}$ is dense in $Exp_{\mathfrak{A}}^{p}(E)$.

We see, as in Proposition 5.1.3, that there exists $\varphi \in Exp_{\mathfrak{A}}^{p}(E)'$ such that $Tf = \varphi * f$ for every $f \in Exp_{\mathfrak{A}}^{p}(E)$. Then $\beta(\varphi) \in \mathcal{H}(p\, B_{E'})$ is not constant. Since $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, $T(e^{\gamma_0})(0) = \beta(\varphi)(\gamma_0)$ is an accumulation point of $\mathbb{T} \setminus D$. Thus, an application of Lemma 5.2.10 proves that the linear span of $\{e^{\gamma} : \|\gamma\| < p,\ \beta(\varphi)(\gamma) \in \mathbb{T} \setminus D\}$ is dense in $Exp_{\mathfrak{A}}^{p}(E)$. ☐

5.3 Frequently hypercyclic subspaces and examples

Finally, we study the existence of frequently hypercyclic subspaces for a given non-trivial convolution operator, that is, the existence of closed infinite-dimensional subspaces in which every non-zero vector is frequently hypercyclic. We prove that there exists a frequently hypercyclic subspace for each non-trivial convolution operator on $\mathcal{H}_{b\mathfrak{A}}(E)$, if the dimension of E is bigger than 1.

Lastly, we study exponential growth conditions for special cases of convolution operators such as translation and partially differentiation ones.

5.3.1 Frequently hypercyclic subspaces

Given a frequently hypercyclic operator T on a Fréchet space X with frequently hypercyclic vector $x \in X$, we can consider the linear subspace $\mathbb{K}[T]x$, whose elements are the evaluations at x of every polynomial on T. It turns out that $\mathbb{K}[T]x \setminus \{0\}$ is contained on $FHC(T)$, the set of all frequently hypercyclic vectors of T, but in general $\mathbb{K}[T]x$ is not closed in X. Then, it is natural to ask if there exists a *closed* subspace $M \subset X$ such that $M \setminus \{0\} \subset FHC(T)$. Bonilla and Grosse-Erdmann, in [BGE12], gave sufficient conditions for this situation to hold. First we state the Frequent Hypercyclicity Criterion.

Theorem 5.3.1 (Frequent Hypercyclicity Criterion)**.** *Let T be an operator on a separable F-space X. Suppose that there exists a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for all $x \in X_0$,*

1. *$\sum_{n=1}^{\infty} T^n x$ converges unconditionally,*
2. *$\sum_{n=1}^{\infty} S^n x$ converges unconditionally,*
3. *$TSx = x$.*

Then T is frequently hypercyclic.

The Bonilla and Grosse-Erdmann theorem for the existence of a frequently hypercyclic subspace states that if an operator T satisfies the Frequent Hypercyclicity Criterion and admits an infinite number of linearly independent eigenvectors, associated to an eigenvalue of modulus less than one then there exists a frequently hypercyclic subspace for T. Since we cannot assure that non-trivial convolution operators satisfy the Frequent Hypercyclicity Criterion, Theorem 5.3.1, we need the following modified version which may be found in [GEPM11, Remark 9.10].

Proposition 5.3.2. *Let T be an operator on a separable F-space X. Suppose that there exists a dense subset X_0 of X and for any $x \in X_0$ there is a sequence $(u_n(x))_{n\geq 0} \subset X$ such that,*

1. *$\sum_{n=1}^{\infty} T^n x$ converges unconditionally,*
2. *$\sum_{n=1}^{\infty} u_n(x)$ converges unconditionally,*
3. *$u_0(x) = x$ and $T^j u_n(x) = u_{n-j}(x)$, for $j \leq n$.*

Then T is frequently hypercyclic.

Now, we can state the modified version of the Bonilla and Grosse-Erdmann theorem which will be used for the proof of Theorem 5.3.4.

Theorem 5.3.3. *Let X be a separable F-space with a continuous norm and T an operator on X that satisfies the hypotheses of Proposition 5.3.2. If $\dim\ker(T-\lambda) = \infty$ for some scalar λ with $|\lambda| < 1$ then T has a frequently hypercyclic subspace.*

The proof of the previous theorem follows the same lines as the proof of [BGE12, Theorem 3], but replacing $S^n y_j$ by $u_n(y_j)$, for each $y_j \in X_0$, in their key Lemma 1. Next, we prove the existence of frequent hypercyclic subspaces for every non-trivial convolution operator, if $\dim(E) > 1$. The corresponding problem for $\dim(E) = 1$ is open, up to our knowledge.

Theorem 5.3.4. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and E a Banach space with $\dim(E) > 1$ and separable dual such that the finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. If $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is a non-trivial convolution operator, then T has a frequently hypercyclic subspace.*

Proof. Let us see that both hypotheses of Theorem 5.3.3 are fulfilled by every non-trivial convolution operator on $\mathcal{H}_{b\mathfrak{A}}(E)$. Recall that if $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is a non-trivial convolution operator then $\beta(\varphi)(\gamma) = T(e^{\gamma})(0)$ is holomorphic as a function of $\gamma \in E'$, and that $T(e^{\gamma}) = [T(e^{\gamma})(0)]e^{\gamma}$. We have that $\{e^{\gamma} : \gamma \in E'\}$ is a linearly independent set in $\mathcal{H}_{b\mathfrak{A}}(E)$, see [AB99, Lemma 2.3]. We will prove that there exists some scalar λ with $|\lambda| < 1$ such that $\dim\ker(T-\lambda) = \infty$. We follow the ideas of the proof of [Pet06, Theorem 5]. If the set of zeros of $\beta(\varphi)$, denoted by $Z(\beta(\varphi)) = \{\gamma \in E' : \beta(\varphi)(\gamma) = 0\}$, is infinite then we take $\lambda = 0$, because $\ker(T) \supset \{e^{\gamma} : \gamma \in Z(\beta(\varphi))\}$. If $Z(\beta(\varphi))$ is not infinite, then it is empty since $\dim(E) > 1$. Now, fix $\gamma \in E'$ and consider $f_{\gamma}(w) = \beta(\varphi)(w\gamma)$ for $w \in \mathbb{C}$. From the continuity of T and of δ_0, we get that there exist positive constants M and s such that

$$|f_\gamma(w)| = |T(e^{w\gamma})(0)| \leq Mp_s(e^{w\gamma}) = M\sum_{k\geq 0}\frac{s^k}{k!}\|d^k(e^{w\gamma})(0)\|_{\mathfrak{A}_k}$$
$$= M\sum_{k\geq 0}\frac{s^k}{k!}\|w\gamma\|^k = Me^{s\|\gamma\||w|}.$$

Thus, $f_\gamma : \mathbb{C} \to \mathbb{C}$ is a holomorphic function of exponential type without zeros. Then there exist complex constants $C(\gamma)$ and $p(\gamma)$ such that $f_\gamma(w) = C(\gamma)e^{p(\gamma)w}$.

Note that $C = C(\gamma)$ is independent of γ because

$$C(\gamma) = f_\gamma(0) = \beta(\varphi)(0) = T(1)(0).$$

We also have that $f'_\gamma(0) = Cp(\gamma) = T(\gamma)(0)$. Thus we get that $p(\gamma) = \frac{1}{C}T(\gamma)(0)$ is a linear continuous functional. Finally, we get that $\beta(\varphi)(\gamma) = Ce^{p(\gamma)}$ with $p \in E''$ and $C \neq 0$. This implies that $Z(\beta(\varphi) - \lambda)$ is infinite for every $\lambda \neq 0$, as we wanted to prove.

To prove that T satisfies the hypotheses of Proposition 5.3.2 we follow the ideas of the second proof of [BGE06, Theorem 1.3]. Parametrizing the eigenvectors e^γ it is possible to construct a family of C^2-functions $C_k : \mathbb{T} \to \mathcal{H}_{b\mathfrak{A}}(E)$ such that $T(C_k(\lambda)) = \lambda C_k(\lambda)$ and such that, for every Borel set of full Lebesgue measure, $B \subset \mathbb{T}$, the linear span of $\{C_k(\lambda) : \lambda \in B,\ k \in \mathbb{N}\}$ is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$. For $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ set

$$x_{k,j} = \int_{\mathbb{T}} \lambda^j C_k(\lambda)d\lambda,$$

where the integral is in the sense of Riemann and $X_0 = \text{span}\{x_{k,j}; j \in \mathbb{Z},\ k \in \mathbb{N}\}$. It follows from the proof of [BG04, Théorème 2.2.] that X_0 is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$ and that for $n \geq 0$, $j \in \mathbb{Z}$, $k \in \mathbb{N}$ we get

$$T^n x_{k,j} = \int_{\mathbb{T}} \lambda^{j+n} C_k(\lambda)d\lambda.$$

For every $y \in X_0$, there exists a linear combination $y = \sum_{l=1}^{m_y} a_l x_{k_l,j_l}$. So, we define

$$u_n(y) = \sum_{l=1}^{m_y} a_l x_{k_l,j_l-n}.$$

Finally, we have that $u_0(y) = y$ and that $T^i u_n(y) = u_{n-i}(y)$ if $i \leq n$, for every $y \in X_0$. Since each C_k is a C^2-function, by [GEPM11, Lemma 9.23 (b)], we obtain that the series $\sum_{n=1}^\infty T^n x_{k,j}$, $\sum_{n=1}^\infty u_n(x_{k,j})$ converge unconditionally for all $j \in \mathbb{Z}$, $k \in \mathbb{N}$. As we claimed, T satisfies the hypotheses of Proposition 5.3.2, and so there exists a frequently hypercyclic subspace. □

5.3.2 Translation operators.

Suppose that $\tau_{z_0} : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is the translation operator defined by $\tau_{z_0}(f)(z) = f(z+z_0)$. The next proposition is similar to [GEPM11, Theorem 9.26], but in this case for translation operators in $\mathcal{H}_{b\mathfrak{A}}(E)$, which gives sharp exponential growth conditions for frequently hypercyclic functions.

Proposition 5.3.5. *Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type and let E be a Banach space with separable dual such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for every k. Suppose that $\mathfrak{A}$ is weakly differentiable with constants $c_{k,l}$ satisfying (4.1.2). Let $\tau_{z_0} : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ be the translation operator by a non-zero vector $z_0 \in E$. Then,*

(a) Given $\varepsilon > 0$, then there exists $C > 0$ and an entire function $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ which is frequently hypercyclic for τ_{z_0} and satisfies

$$|f(z)| \leq Ce^{\varepsilon\|z\|}.$$

(b) Let $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\liminf_{r\to\infty} \varepsilon(r) = 0$ and C any positive number. Then there is no frequently hypercyclic entire function $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ for τ_{z_0}, satisfying

$$|f(z)| \leq Ce^{\varepsilon(\|z\|)\|z\|}, \textit{ for all } z.$$

Proof. (a) Note that $\tau_{z_0}(e^\gamma) = e^{\gamma(z_0)}e^\gamma$, thus

$$\inf\{\|\gamma\|, \gamma \in E' \text{ such that } |\tau_{z_0}(e^\gamma)(0)| = 1\} = 0.$$

It follows from Theorem 5.2.11 that for any $\varepsilon > 0$, there exist a frequently hypercyclic function $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ such that

$$|f(z)| \leq Ce^{\varepsilon\|z\|},$$

for some positive constant C.

(b) Suppose that there exist a frequently hypercyclic function f for τ_{z_0} such that $|f(z)| \leq Ce^{\varepsilon(\|z\|)\|z\|}$. Consider the complex line $L = \{\lambda z_0, \lambda \in \mathbb{C}\}$ and the restriction map given by

$$\begin{aligned} \mathcal{H}_{b\mathfrak{A}}(E) &\longrightarrow \mathcal{H}(\mathbb{C}) \\ g &\mapsto g|_L(\lambda) = g(\lambda z_0). \end{aligned}$$

Consider the following diagram

$$\begin{tikzcd} \mathcal{H}_{b\mathfrak{A}}(E) \arrow[r, "\tau_{z_0}"] \arrow[d] & \mathcal{H}_{b\mathfrak{A}}(E) \arrow[d] \\ \mathcal{H}(\mathbb{C}) \arrow[r, "\tau_1"'] & \mathcal{H}(\mathbb{C}) \end{tikzcd}$$

Note that is a commutative diagram, for $g \in \mathcal{H}_{b\mathfrak{A}}(E)$

$$(\tau_{z_0}g)|_L(\lambda) = \tau_{z_0}g(\lambda z_0) = g((\lambda+1)z_0) = (g|_L)(\lambda+1) = \tau_1(g|_L)(\lambda).$$

Also the restriction map has dense range: take $\gamma \in E'$ such that $\gamma(z_0) = 1$, then $\gamma^k|_L(\lambda) = \gamma^k(\lambda z_0) = \lambda^k$. Thus, all polynomials belong to the range of the restriction map.

Applying the hypercyclic comparison principle we get that τ_1 is frequent hypercyclic and that $f|_L \in \mathcal{H}(\mathbb{C})$ is a frequently hypercyclic function that satisfies

$$|f|_L(z)| = |f(\lambda z_0)| \leq Ce^{\varepsilon(\|\lambda z_0\|)\|\lambda z_0\|}.$$

But this bound contradicts [GEPM11, Theorem 9.26], which states that there is no such a function in $\mathcal{H}(\mathbb{C})$. ☐

Remark 5.3.6. As we mentioned in the proof of the last proposition, in [BBGE10, GEPM11] it is proved that, given ε such that $\liminf_{\lambda\to\infty} \varepsilon(|\lambda|) = 0$, there are no frequently hypercyclic functions for the translation operator in $\mathcal{H}(\mathbb{C})$ satisfying that $|f(\lambda)| \leq Ce^{\varepsilon(|\lambda|)|\lambda|}$. In contrast, there are hypercyclic functions of arbitrary slow growth (see [DR83]). The corresponding result in the Banach space setting has not been studied, up to our knowledge.

5.3.3 Differentiation operators.

For the differentiation operator on $\mathcal{H}_{b\mathfrak{A}}(E)$, $\mathcal{D}_a : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$, $\mathcal{D}_a(f) = d^1 f(\cdot)(a)$, we can estimate the exponential type for the frequent hypercyclic functions. Since $\mathcal{D}_a(e^\gamma)(0) = d^1(e^\gamma)(0)(a) = \gamma(a)$, we get that

$$\inf\{\|\gamma\|, \gamma \in E' \text{ such that } |\mathcal{D}_a(e^\gamma)(0)| = 1\} = \|a\|.$$

Thus given $\varepsilon > 0$ there exist a frequently hypercyclic function f such that

$$|f(x)| \leq Ce^{(\|a\|+\varepsilon)\|x\|},$$

for some $C > 0$. It is not difficult to see that the best exponential type of a hypercyclic function for $\mathcal{D}_a$ is $\|a\|$. To prove this fact it suffices to conjugate $\mathcal{D}_a$ by the one dimensional differentiation operator (as we did in the proof of Proposition 5.3.5) and apply [GEPM11, Theorem 4.22] (see also [GE90] and [Shk93]).

Chapter 6

Non-convolution hypercyclic operators on spaces of holomorphic functions on Banach spaces

In this chapter we study the hypercyclic behavior of non-convolution operators defined on spaces of holomorphic functions over Banach spaces. The operators in the family we analyze are a composition of differentiation and composition operators and are analogues to the operators that we studied in Chapter 3. The hypercyclic behavior varies in terms of several parameters involved. We also prove a Runge type theorem for holomorphic functions on Banach spaces. The results of this chapter are included in [MPSb].

First we define the family of operators that we will study. We prove that they are well defined and bounded. Also we prove that under suitable hypothesis the linear span of the monomials maps is dense in the space of holomorphic functions, which we will need in order to apply the Hypercyclic Criterion.

Then we prove our main result about the hypercyclic behavior of the operators in the family that concerns us. We need an auxiliary result similar to Runge's approximation theorem for holomorphic functions on Banach spaces. Also we study the hypercyclic behavior in the particular space $H_{bc}(E)$, of entire functions of compact type.

6.1 Non-convolution operators in $\mathcal{H}_{b\mathfrak{A}}(E)$

In Chapter 4, we defined the spaces of holomorphic functions on a Banach space. We may also define a space of entire functions on $B(x,r)$ of bounded $\mathfrak{A}$-type [CDM07, FJ09] as the set of entire functions with $\mathfrak{A}$-radius of convergence at x greater that r.

Definition 6.1.1. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type, E be a Banach space, $x \in E$, and $r > 0$. We define the space of holomorphic functions of $\mathfrak{A}$-bounded type on $B(x,r)$ by

$$\mathcal{H}_{b\mathfrak{A}}(B(x,r)) = \left\{ f \in H(B(x,r)) \ : d^k f(x) \in \mathfrak{A}_k(E) \text{ and } \limsup_{k\to\infty} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathfrak{A}_k}^{1/k} \leq \frac{1}{r} \right\}.$$

We consider in $\mathcal{H}_{b\mathfrak{A}}(B(x,r))$ the seminorms p_t^x, for $0<t<r$, given by

$$p_t^x(f)=\sum_{k=0}^{\infty}\Big\|\frac{d^kf(x)}{k!}\Big\|_{\mathfrak{A}_k}t^k,$$

for all $f\in\mathcal{H}_{b\mathfrak{A}}(B(x,r))$. It is easy to show that $\Big(\mathcal{H}_{b\mathfrak{A}}(B(x,r)),\{p_t^x\}_{0<t<r}\Big)$ is a Fréchet space.

Our objective is to define analogues of the operators we studied for holomorphic functions of finite variables in [MPS14] and to determine the dynamics they induce. It is clear that $\mathcal{H}_{b\mathfrak{A}}(E)$ must be separable in order to support an hypercyclic operator. Since E' is a subspace of $\mathcal{H}_{b\mathfrak{A}}(E)$, we need to restrict ourselves to the class of Banach spaces with separable dual space. On the other hand, if E' is separable and if we assume that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$, it is a simple exercise to prove that $\mathcal{H}_{b\mathfrak{A}}(E)$ is separable. Also, in order to be able to define directional derivatives we will assume that the space E has an unconditional basis. Recall the following definition from [GM93, Theorem 1].

Definition 6.1.2. A basis $(e_s)_{s\in\mathbb{N}}$ is C-unconditional if there exist a positive constant C such that for every sequence of scalars $(a_n)_{n\in\mathbb{N}}$ and every sequence of scalars $(\varepsilon_n)_{n\in\mathbb{N}}$ of modulus at most 1, we have the inequality

$$\|\sum_{n=1}^{\infty}\varepsilon_na_ne_n\|\le C\|\sum_{n=1}^{\infty}a_ne_n\|.$$

Definition 6.1.3. Let E be a Banach space with basis $(e_s)_{s\in\mathbb{N}}$. We say that the basis is shrinking if for every $e'\in E'$ the norm of the restriction of e' to the span of $(e_s)_{s\ge n}$, $e'|_{[e_s]_{s\ge n}}$, goes to 0 as $n\to\infty$.

Let E be a Banach space with a unconditional basis $(e_s)_{s\in\mathbb{N}}$. The dual system of linear functionals associated to the basis $(e_s)_{s\in\mathbb{N}}$ is defined by the relation $e'_m(e_n)=\delta_{m,n}$. Let $\{P_s\}_{s\in\mathbb{N}}$ be the natural projections associated to the basis $(e_s)_{s\in\mathbb{N}}$. For every choice of scalars $(a_s)_{s\in\mathbb{N}}$ and for all integers $n<m$ we have $P_n^*(\sum_{s\le m}a_se'_s)=\sum_{s\le n}a_se'_s$. Since, $\|P_n^*\|=\|P_n\|$, we get that $(e'_s)_{s\in\mathbb{N}}$ is a basic sequence in E', whose basis constant is identical to that of $(e_s)_{s\in\mathbb{N}}$. Recall that $(e'_s)_{s\in\mathbb{N}}$ form a basis of E' if and only if $(e_s)_{s\in\mathbb{N}}$ is shrinking [JL79, Proposition 1.b.1].

Definition 6.1.4. If $\beta=(\beta_i)_{i\in\mathbb{N}}$ is a finite multi-index, we define the monomial $z^\beta\in\mathcal{H}_{b\mathfrak{A}}(E)$ as

$$z^\beta=\prod_i(e'_i)^{\beta_i}.$$

In order to apply the hypercyclicity criterion, we will need a convenient dense subset. In general, we will use the span of the monomials. If E has a unconditional basis and E' is separable, then the basis is shrinking, because $\ell_1\nsubseteq E$. The next lemma tell us that under suitable assumptions on the holomorphy type, the monomials span a dense set in $\mathcal{H}_{b\mathfrak{A}}(E)$.

Definition 6.1.5. We say that the holomorphy type $\mathfrak{A}=\{\mathfrak{A}_k\}_{k=0}^{\infty}$ is coherent if for each $k\ge0$ there exist a positive constant d_k such that for every Banach space E, the following hold:

$$\text{if } P\in\mathfrak{A}_k(E),\ \gamma\in E' \text{ then } \gamma P \text{ belongs to } \mathfrak{A}_{k+1}(E) \text{ and } \|\gamma P\|_{\mathfrak{A}_{k+1}(E)}\le d_k\|P\|_{\mathfrak{A}_k(E)}\|\gamma\|_{E'}. \tag{6.1.1}$$

Lemma 6.1.6. Let E be a Banach space with unconditional shrinking basis $(e_s)_{s\in\mathbb{N}}$ and let $\mathfrak{A}$ be a coherent holomorphy type such that finite type polynomials are dense in $\mathfrak{A}_k(E)$ for each k. Then, the linear span of the monomials is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$.

Proof. Since $\mathfrak{A}$ is coherent, if $\varphi^1, \dots, \varphi^n \in E'$, we get that

$$\begin{aligned}
\|\varphi^1 \dots \varphi^n\|_{\mathfrak{A}_n(E)} &\leq d_{n-1}\|\varphi^1\|_{\mathfrak{A}_1(E)}\|\varphi^2 \dots \varphi^n\|_{\mathfrak{A}_{n-1}(E)} \\
&\leq \left(\prod_{j=1}^{n-1} d_j\right) \prod_{j=1}^{n} \|\varphi^j\|_{E'}.
\end{aligned}$$

Since the basis of E is shrinking and $(e'_s)_{s\in\mathbb{N}}$ form a basis of E', we get that each $\varphi^j \in E'$ can be written as

$$\varphi^j = \sum_{s=1}^{\infty} a_s^{(j)} e'_s.$$

Now, given $\varepsilon > 0$ and $N > 0$, we fix $\varepsilon_i < \frac{\varepsilon}{nK^{i-1}\prod_{j=1}^{n-1} d_j \prod_{j\neq i}\|\varphi^j\|_{E'}}$ and $\xi^j := \sum_{s=1}^{N} a_s^{(j)} e'_s \in E'$ such that $\|\varphi^j - \xi^j\|_{E'} < \varepsilon_j$.

Note that $\prod_{j=1}^{n} \xi^j$ is a linear combination of monomials because,

$$\prod_{j=1}^{n} \xi^j = \prod_{j=1}^{n}\sum_{s=1}^{N} a_s^{(j)} e'_s = \sum_{s_1,\dots,s_n=0}^{N} a_{s_1}^{(1)} \dots a_{s_n}^{(n)} \prod_{j=1}^{n} e'_{s_j}.$$

Also, since $(e'_s)_{s\in\mathbb{N}}$ is a basis, there exist a constant K such that

$$\|\xi^j\|_{E'} = \left\|\sum_{s=1}^{N} a_s^{(j)} e'_s\right\|_{E'} \leq K\|\varphi^j\|_{E'}.$$

Thus, we have

$$\begin{aligned}
\|\varphi^1 \dots \varphi^n - \xi^1 \dots \xi^n\|_{\mathfrak{A}_n(E)} &\leq \sum_{i=1}^{n} \left\|\left(\prod_{j=1}^{i-1} \xi^j\right)\left(\prod_{j=i}^{n} \varphi^j\right) - \left(\prod_{j=1}^{i} \xi^j\right)\left(\prod_{j=i+1}^{n} \varphi^j\right)\right\|_{\mathfrak{A}_n(E)} \\
&= \sum_{i=1}^{n} \left\|\left(\prod_{j=1}^{i-1} \xi^j\right)(\varphi^i - \xi^i)\left(\prod_{j=i+1}^{n} \varphi^j\right)\right\|_{\mathfrak{A}_n(E)} \\
&\leq \sum_{i=1}^{n}\left(\prod_{j=1}^{n-1} d_j\right)\left(\prod_{j=1}^{i-1}\|\xi^j\|_{E'}\right)\|\varphi^i - \xi^i\|_{E'}\left(\prod_{j=i+1}^{n}\|\varphi^j\|_{E'}\right) \\
&\leq \sum_{i=1}^{n}\left(\prod_{j=1}^{n-1} d_j\right) K^{i-1}\left(\prod_{j\neq i}\|\varphi^j\|_{E'}\right)\|\varphi^i - \xi^i\|_{E'}
\end{aligned}$$

So, we get that $\|\varphi^1 \dots \varphi^n - \xi^1 \dots \xi^n\|_{\mathfrak{A}_n(E)} < \varepsilon$. ☐

Remark 6.1.7. Reciprocally, suppose that the linear span of the monomials of degree k is dense in $\mathfrak{A}_k(E)$, for all $k \in \mathbb{N}$. Since the norm of $\mathfrak{A}_1(E)$ coincides with the norm in E', the linear span of the monomials of degree 1 is dense in E'. But, this means that $(e'_s)_{s\in\mathbb{N}}$ is a basis of E'.

There are other examples of spaces of holomorphic functions in which the span of the monomials is dense. For example, in $H_b(c_0)$ or $H_b(T^*)$ where T^* is the Tsirelson space. Also, is E is asplund and $\mathfrak{A}$ is the sequence of ideals of extendible polynomials, then the monomials span a dense subset in $\mathcal{H}_{b\mathfrak{A}}(E)$, see [CG11, Corollary 2.5].

Now we define the family of operators we will study and prove that they are bounded on $\mathcal{H}_{b\mathfrak{A}}(E)$. Let E be a Banach space with a C-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Fix a finite multi-index $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, $|\alpha| = m$, which counts how many times the operator T partially differentiates in each variable, where the partial derivative in the $s-$th variable is

$$D^{e_s} f(z) = \lim_{h\to 0} \frac{f(z+he_s) - f(z)}{h}.$$

Also fix two sequences, $\lambda = (\lambda_j)_{j\in\mathbb{N}} \in \ell_\infty$ and $b = \sum_{j\in\mathbb{N}} b_j e_j \in E$. The operator $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is defined by

$$Tf(z) = D^\alpha f(\lambda z + b). \tag{6.1.2}$$

Proposition 6.1.8. Let $\mathfrak{A}$ is a holomorphy type with constants as in (4.1.2), and T defined as in (6.1.2), then T is a continuous linear operator on $\mathcal{H}_{b\mathfrak{A}}(E)$. Moreover, for each $f \in \mathcal{H}_{b\mathfrak{A}}(E)$, $x \in E$, and $r, \varepsilon > 0$,

$$p_r^x(Tf) \leq \frac{C(\alpha)}{\varepsilon^{|\alpha|}} p^{\lambda\cdot x+b}_{rC\|\lambda\|_\infty+\varepsilon}(f), \tag{6.1.3}$$

where $C(\alpha)$ is a positive constant depending only on α, which can be taken equal to $e^{|\alpha|+1}(\prod_{\alpha_i\neq 0}\alpha_i)^{1/2}$.

Observe that we can think T as a composition of three operators. Indeed, let $\Lambda : x \mapsto \lambda \cdot x$ be the coordinate-wise multiplication operator on E, which satisfies $\|\Lambda\| \leq C\|\lambda\|_\infty$. Then Λ induces a composition operator $M_\lambda : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$, defined by $M_\lambda(f) = f \circ \Lambda$. Then,

$$Tf = M_\lambda \circ \tau_b \circ D^\alpha(f),$$

where $\tau_b : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ is the translation operator defined by $\tau_b(f)(z) = f(z + b)$.

To prove the above proposition we will show that the three operators are continuous on $\mathcal{H}_{b\mathfrak{A}}(E)$. For the partial differentiation operator D^α we will need two lemmas. The first one, which should be well known, shows that it coincides with the differentials forms and the second is a generalization of the Cauchy inequalities to holomorphy types.

Lemma 6.1.9. Let E be a Banach space with basis $(e_n)_n$ and let $f \in H(E)$ be a holomorphic function on E. Then

(i) $D^{e_s} f(z) = df(z)(e_s)$,

(ii) $d[d^k f(\cdot)^\vee(e_{s_1}, \ldots, e_{s_k})](z)(e_l) = [d^{k+1} f(z)]^\vee(e_l, e_{s_1}, \ldots, e_{s_k})$.

(iii) $D^\alpha f(z) = [d^{|\alpha|} f(z)]^\vee(e_1^{\alpha_1}, \ldots, e_l^{\alpha_l})$, if $\alpha_j = 0$ for every $j > l$.

Proof. To prove (i), we write $f(w) = \sum_{k\geq 0} \frac{d^k f(z)}{k!}(w-z)$. Thus we have that

$$\begin{aligned}\frac{f(z+he_s)-f(z)}{h} &= \frac{1}{h}\sum_{k\geq 1}\frac{d^k f(z)}{k!}(he_s) = \sum_{k\geq 1} h^{k-1}\frac{d^k f(z)}{k!}(e_s)\\ &= df(z)(e_s) + h\left[\sum_{k\geq 2} h^{k-2}\frac{d^k f(z)}{k!}(e_s)\right] \underset{h\to 0}{\longrightarrow} df(z)(e_s).\end{aligned}$$

The last assertion is true because since $f \in H(E)$, it satisfies that $\limsup \left\|\frac{d^k f(z)}{k!}\right\|^{1/k} < \infty$, and thus $\sum_{k\geq 2} h^{k-2}\frac{d^k f(z)}{k!}(e_s)$ is a power series with positive radius of convergence.

For (ii), observe that if $z \in E$,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{d^k f(a)}{k!}(x) = f(x+a) &= \sum_{j=0}^{\infty}\frac{d^j f(z)}{j!}(x+a-z)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}\left[\frac{d^j f(z)}{j!}\right]^{\vee}((a-z)^{j-k},x^k)\\ &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}\left[\frac{d^j f(z)}{j!}\right]^{\vee}((a-z)^{j-k},x^k).\end{aligned}$$

Thus we get that

$$d^k f(a)(x) = k!\sum_{j=1}^{\infty}\binom{j}{k}\left[\frac{d^j f(z)}{j!}\right]^{\vee}((a-z)^{j-k},x^k),$$

from which we conclude that if $g(z) = d^k f(z)^{\vee}(e_{s_1},\ldots,e_{s_k})$, then

$$\begin{aligned}\frac{g(z+he_l)-g(z)}{h} &= \frac{1}{h}\left[k!\sum_{j=k}^{\infty}\binom{j}{k}\left[\frac{d^j f(z)}{j!}\right]^{\vee}((he_l)^{j-k},e_{s_1},\ldots,e_{s_k}) - d^k f(z)^{\vee}(e_{s_1},\ldots,e_{s_k})\right]\\ &= \frac{1}{h}\left[k!\sum_{j=k+1}^{\infty}\binom{j}{k}h^{j-k}\left[\frac{d^k f(z)}{k!}\right]^{\vee}(e_l^{j-k},e_{s_1},\ldots,e_{s_k})\right]\\ &\underset{h\to 0}{\longrightarrow} \quad [d^{k+1}f(z)]^{\vee}(e_l,e_{s_1},\ldots,e_{s_k}),\end{aligned}$$

because f has positive radius of convergence at z.

(iii) follows from (i) and (ii). ☐

Using the previous lemma it is immediate to translate a result in [Mur12, Lemma 4.5] to conclude that D^α is a bounded operator on $\mathcal{H}_{b\mathfrak{A}}(E)$.

Lemma 6.1.10 (Cauchy estimates for holomorphy types)**.** *Let $\mathfrak{A}$ be a holomorphy type with constants as in* (4.1.2)*, E be a Banach space with basis $(e_n)_n$. Then for $f \in \mathcal{H}_{b\mathfrak{A}}(E)$, $x \in E$ and $r, \varepsilon > 0$, we have*

$$p_r^x(D^\alpha f) \leq \frac{C(\alpha)}{\varepsilon^{|\alpha|}}p_{r+\varepsilon}^x(f),$$

where $C(\alpha)$ is a positive constant depending only on α, we can take $C(\alpha)$ to be equal to $e^{|\alpha|+1}(\prod_{\alpha_i\neq 0}\alpha_i)^{1/2}$.

Proof of Proposition 6.1.8. Let us first show estimates for M_λ in terms of the seminorms p_r^x. If $f=\sum_k P_k$ then

$$\|M_\lambda(P_k)\|_{\mathfrak{A}_k(E)} = \|P_k\circ\Lambda\|_{\mathfrak{A}_k(E)} \leq \|P_k\|_{\mathfrak{A}_k(E)}(C\|\lambda\|_\infty)^k,$$

and

$$\frac{d^j(M_\lambda f)(x)}{j!} = \sum_{k\geq j}\binom{k}{j}(M_\lambda P_k)_{x^{k-j}}.$$

Thus, using that $\mathfrak{A}_j$ is an ideal, we have

$$\begin{aligned}
p_r^x(M_\lambda f) &= \sum_{j\geq 0} r^j \left\|\frac{d^j(M_\lambda f)(x)}{j!}\right\|_{\mathfrak{A}_j(E)} = \sum_{j\geq 0} r^j \left\|\sum_{k\geq j}\binom{k}{j}(M_\lambda P_k)_{x^{k-j}}\right\|_{\mathfrak{A}_j(E)} \\
&= \sum_{j\geq 0} r^j \left\|M_\lambda\left(\sum_{k\geq j}\binom{k}{j}(P_k)_{\Lambda(x)^{k-j}}\right)\right\|_{\mathfrak{A}_j(E)} \\
&\leq \sum_{j\geq 0}(rC\|\lambda\|_\infty)^j \left\|\sum_{k\geq j}\binom{k}{j}(P_k)_{(\Lambda(x))^{k-j}}\right\|_{\mathfrak{A}_j(E)} = p^{\Lambda(x)}_{rC\|\lambda\|_\infty}(f).
\end{aligned}$$

For the translation operator we have,

$$\begin{aligned}
p_r^x(\tau_b f) &= \sum_{j\geq 0}\left\|\frac{d^j(\tau_b f)(x)}{j!}\right\|_{\mathfrak{A}_j(E)} r^j \\
&= \sum_{j\geq 0} r^j\left\|\frac{d^j f(x+b)}{j!}\right\|_{\mathfrak{A}_j(E)} = p_r^{x+b}(f).
\end{aligned}$$

To finish the proof just apply the Cauchy inequalities (Lemma 6.1.10) together with the estimates for M_λ and τ_b. $\square$

6.2 Hypercyclic behavior of the operator

In this section we take care of the hypercyclic behavior of the operators in this family. If $\lambda_j=0$ for some j, then we have that $d(T^n f)(\cdot)(e_j)=0$, for every $n\in\mathbb{N}$. Since, the application $dg(\cdot)(e_j)$ is continuous, we conclude that the orbit of f under T can not be dense.

The next result describes the hypercyclicity of the operator $Tf = M_\lambda\circ\tau_b\circ D^\alpha(f)$, with $\lambda_i\neq 0$ for all $i\in\mathbb{N}$ and $\alpha\neq 0$, in terms of the parameters involved. Let us denote $\lambda^\alpha=\prod_i\lambda_i^{\alpha_i}$. When no coordinate of the map ϕ is a translation, we denote $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3),\dots)$ the sequence in $\mathbb{C}^{\mathbb{N}}$ formed by the fixed points of every coordinate of the map ϕ. It is worth to notice that if $b_i=0$ and $\lambda_i=1$, then the fixed point of the i-coordinate of ϕ is 0, thus we suppose that $0/0=0$. Our main theorem reads as follows.

Theorem 6.2.1. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$, with constants as in (4.1.2). Let T be the operator on $\mathcal{H}_{b\mathfrak{A}}(E)$, defined by $Tf(z) = M_\lambda \circ \tau_b \circ D^\alpha f(z)$, with $\alpha \neq 0$ and $\lambda_i \neq 0$ for all $i \in \mathbb{N}$.

a) If $|\lambda^\alpha| \geq 1$, then T is strongly mixing in the gaussian sense.

b) If $\|\lambda\|_\infty = 1$ and $b_i \neq 0$ and $\lambda_i = 1$ for some $i \in \mathbb{N}$, then T is mixing.

c) If $\|\lambda\|_\infty = 1$, no coordinate of ϕ is a translation and $\zeta \notin E''$, then T is mixing.

d) If $|\lambda^\alpha| < 1$ and $\zeta \in E''$, then T is not hypercyclic.

We will divide the proof in several cases. The first cases we will prove are those in which $|\lambda^\alpha| \geq 1$. Let $A := \{n \in \mathbb{N} : \lambda_n = 1\}$ and $B := \{n \in \mathbb{N} : \lambda_n \neq 1\}$. If $w \in \mathbb{C}^{\mathbb{N}}$, we write $w_A = (w_i)_{i\in A}$ and $w_B = (w_i)_{i\in B}$. We have that $\mathbb{N} = A\dot{\cup}B$. We can also decompose $E = E(A) + E(B)$. We will show that the conditions of the hypercyclicity criteria are satisfied with dense subspaces of the form $span\{e_\gamma z^\beta : \gamma \in E', \gamma_B = 0, \beta_A = 0\}$. Since the basis $(e_s)_{s\in\mathbb{N}}$ is shrinking, we can think of the elements of E' as sequences in $(e'_s)_{s\in\mathbb{N}}$, the dual system of the basis of E. The vectors γ and β only have finite non-zero coordinates.

Lemma 6.2.2. Let E be a Banach space with a unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a coherent holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Suppose that $|\lambda^\alpha| \geq 1$ and $\alpha_B = 0$. Then T is strongly mixing in the gaussian sense.

Proof. We show that T satisfies the conditions of Theorem 1.3.9. Take a function of the form $e_\gamma z^\beta$, with $\gamma_B = 0$ and $\beta_A = 0$. Define $c_\beta \in E$ as, $c_\beta(n) = 0$ if $\beta_n = 0$, and $c_\beta(n) = \frac{b_n}{\lambda_n - 1}$ if $\beta_n \neq 0$ and define τ_β as the translation operator by c_β (note that c_β has finite non zero coordinates). Then $\tau_\beta^{-1} \circ T \circ \tau_\beta(e_\gamma z^\beta) = \gamma^\alpha e^{\langle \gamma_A, b_A\rangle} \lambda^\beta e_\gamma z^\beta$. Therefore,

$$T(\tau_\beta e_\gamma z^\beta) = \gamma^\alpha e^{\langle \gamma_A, b_A\rangle} \lambda^\beta \, \tau_\beta e_\gamma z^\beta,$$

that is, the functions $\tau_\beta e_\gamma z^\beta$ are eigenvectors of T.

Let $D \subset \mathbb{S}^1$ a dense subset. It is enough to prove that

$$span\{\tau_\beta e_\gamma z^\beta : \text{ with } \gamma_B = 0, \beta_A = 0, \gamma^\alpha e^{\langle \gamma_A, b_A\rangle} \lambda^\beta \in D\}$$

is dense in $H_{b\mathfrak{A}}(E)$. Define $f_\beta(\gamma) = \gamma^\alpha e^{\langle \gamma_A, b_A\rangle} \lambda^\beta$. For each β finite with $\beta_A = 0$, the function f_β is holomorphic on $E(A)'$ and not constant. By [MPS14, Lemma 2.4], we get that $\{e_\gamma : f_\beta(\gamma) \in D\}$ span a dense subspace in $H_{b\mathfrak{A}}(E(A))$, for each β finite with $\beta_A = 0$. Also, note that for $k \in \mathbb{N}_0$

$$span\{\tau_\beta(z^\beta) : |\beta| \leq k\} = span\{z^\beta : |\beta| \leq k\}.$$

This is clear for $k = 0$, because both sets are $\mathbb{C}$, and if $|\beta| = k$

$$(z - c_\beta)^\beta = \prod_i \sum_{j=0}^{\beta_i} {z_i}^{\beta_i - j} {c_{\beta}}_i^{\beta_i} \binom{\beta_i}{j} = z^\beta + g(z),$$

where g has monomials of degree $< k$. Also, note that $\tau_{\beta_B}(z^{\beta_A} z^{\beta_B}) = z^{\beta_A} \tau_{\beta_B}(z^{\beta_B})$, and so $span\{\tau_{\beta_B}(z^{\beta_A} z^{\beta_B})\}$ is dense in $H_{b\mathfrak{A}}(E)$, because by Lemma 6.1.6, the monomials span a dense subspace of $\mathcal{H}_{b\mathfrak{A}}(E)$. Gathering the previous observations we get that the eigenvectors of T with eigenvalues in D span a dense subspace in $\mathcal{H}_{b\mathfrak{A}}(E)$. Thus, we have seen that the conditions of Theorem 1.3.9 are satisfied, and so the operator T is strongly mixing in the gaussian sense. $\square$

It remains to prove the case when $|\lambda^\alpha| \geq 1$ and T differentiates in some coordinate with $\lambda_n \neq 1$.

Lemma 6.2.3. Let E be a Banach space with a unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a coherent holomorphy type such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Suppose that $|\lambda^\alpha| \geq 1$ and $\alpha_B \neq 0$. Then T is strongly mixing in the gaussian sense.

Proof. We will show that T satisfies the conditions of Theorem 1.3.10. Let $D := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n \neq 0\}$, note that D is a finite set. Then T is topologically conjugate to

$$T_0 f(z) = D^\alpha f(\lambda z + \tilde{b})$$

through a translation, where $\tilde{b}_n = b_n$ for all $n \notin D$ and $\tilde{b}_n = 0$ for all $n \in D$. Indeed, defining $c \in E$ as $c_n = 0$ for $n \notin D$ and $c_n = \frac{-b_n}{\lambda_n - 1}$ for $n \in D$, we get that $T_0 \circ \tau_c = \tau_c \circ T$. We may thus assume that $b_n = 0$ for every n such that $\lambda_n \neq 1$ and $\alpha_n \neq 0$. So we can split $\mathbb{N}$ into three disjoint sets,

$$A := \{n \in \mathbb{N} : \lambda_n = 1\},$$

$$C := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n = 0\},$$

$$D := \{n \in \mathbb{N} : \lambda_n \neq 1, \alpha_n \neq 0\}.$$

Note that $|\lambda^\alpha| = |\lambda_D^{\alpha_D}| \geq 1$. Define the subspace

$$X_0 = span\{e_\gamma z^\beta : \gamma \in E',\, \gamma_D = \gamma_C = 0,\, \beta_A = 0\}.$$

Similarly to the above we can see that X_0 is dense in $\mathcal{H}_{b\mathfrak{A}}(E)$. We have that

$$T(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = {\gamma_A}^{\alpha_A} e^{\langle \gamma_A, b_A\rangle} e_{\gamma_A} \frac{\beta_D!}{(\beta_D - \alpha_D)!} z^{\beta_D - \alpha_D} \lambda_D^{\beta_D - \alpha_D} (\lambda z + b)^{\beta_C}.$$

Denote $L(z) = \lambda z + b$, then we can write $(\lambda z + b)^{\beta_C} = C_L(z^{\beta_C})$ with C_L the composition operator associated to L. We also have

$$T^n(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = {\gamma_A}^{n\alpha_A} e^{n\langle \gamma_A, b_A\rangle} e_{\gamma_A} \frac{\beta_D!}{(\beta_D - n\alpha_D)!} z^{\beta_D - n\alpha_D} \lambda_D^{n\beta_D - \frac{n(n-1)}{2}\alpha_D} {C_L}^n(z^{\beta_C}).$$

Then $\sum_n T^n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ is unconditionally convergent because it is a finite sum.

Define a sequence of operators S_n on X_0 by

$$S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C}) = \frac{\beta_D!}{{\gamma_A}^{n\alpha_A} e^{n\langle \gamma_A, b_A\rangle} (\beta_D + n\alpha_D)! \lambda_D^{n\beta_D + \frac{n(n+1)}{2}\alpha_D}} e_{\gamma_A} z^{\beta_D + n\alpha_D} (C_{L^{-1}})^n (z^{\beta_C}),$$

where $L^{-1}(z) = \frac{z-b}{\lambda}$. The operators S_n are defined so that they satisfy $T \circ S_1 = Id$ and $T \circ S_n = S_{n-1}$ on X_0.

Observe that, if $\|z\|_E \leq R$

$$|{C_{L^{-1}}}^n(z^{\beta_C})| \leq \left(\frac{1}{|{\lambda_C}^{\beta_C}|}\right)^n \left(\|z\|_E + \|b_C\|_E \frac{|\lambda_C|^n + 1}{|\lambda_C - 1|}\right)^{\beta_C} \leq M^{n|\beta|} \frac{R^{|\beta|}}{|\lambda_C^{\beta_C}|^n},$$

where M is a positive constant depending only on λ_C and b_C. Thus, since $|\lambda_D{}^{\alpha_D}|^{n(n+1)/2} \geq 1$, we have

$$|S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})| \leq \frac{K^n}{(\beta_D + n\alpha_D)!} \frac{1}{|\lambda_C^{\beta_C}|^n}.$$

Since $S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ depends only on finite variables, this implies unconditionally convergence in $\mathcal{H}_{b\mathfrak{A}}(E)$. In fact, suppose that $Q \in \mathfrak{A}_k(E)$ depends only on N variables and consider the following diagram

$$\begin{tikzcd} E \arrow[r, "Q"] \arrow[d, "\pi"'] & \mathbb{C} \\ \mathbb{C}^N \arrow[ur, "\widetilde{Q}"'] & \end{tikzcd}$$

Since $\mathfrak{A}$ is a Banach ideal, we get that

$$\|Q\|_{\mathfrak{A}_k(E)} \leq \|\widetilde{Q}\|_{\mathfrak{A}_k(\mathbb{C}^N)} \|\pi\|^k \leq N^k \|\pi\|^k \|\widetilde{Q}\|_\infty = N^k \|\pi\|^k \|Q\|_\infty.$$

Then $\sum_n S_n(e_{\gamma_A} z^{\beta_D} z^{\beta_C})$ is unconditionally convergent in $\mathcal{H}_{b\mathfrak{A}}(E)$. Thus, we have proved that T satisfies the conditions of Theorem 1.3.10. □

The two previous lemmas prove that T is strongly mixing in the gaussian sense, in the case $|\lambda^\alpha| \geq 1$. In order to study the hypercyclicity of the operator when $|\lambda^\alpha| < 1$, we need a version of Runge's Theorem for the space $\mathcal{H}_{b\mathfrak{A}}(E)$. Recall the classical Runge's approximation theorem [BM09, Appendix A].

Theorem 6.2.4. [Runge's approximation theorem] Let K be a compact subset of $\mathbb{C}$, and assume that $\mathbb{C} \setminus K$ is connected. Then, any function that is holomorphic in a neighbourhood of K can be uniformly approximated on K by polynomial functions.

In [Hal07], A. Hallack proves that the translations are hypercyclic in the space $H_{bc}(E)$ of entire functions of compact bounded type. For that, the following version of Runge's approximation theorem is proved.

Theorem 6.2.5. Let B_1 and B_2 be two disjoint closed balls in a complex Banach space E. If f is a holomorphic and bounded function in a uniform neighborhood of $B_1 \cup B_2$, then f can be uniformly approximated by polynomials on $B_1 \cup B_2$.

In order to obtain a result in this direction for the space $\mathcal{H}_{b\mathfrak{A}}(E)$, we need to work with multiplicative holomorphy types, which is a slightly more restrictive definition than a coherent sequence.

Definition 6.2.6. Let $\{\mathfrak{A}_k\}_k$ be a sequence of polynomial ideals. We say that $\{\mathfrak{A}_k\}_k$ is multiplicative at E if there exist constants $c_{k,l} > 0$ such that for each $P \in \mathfrak{A}_k(E)$ and $Q \in \mathfrak{A}_l(E)$, we have that $PQ \in \mathfrak{A}_{k+l}(E)$ and

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq c_{k,l} \|P\|_{\mathfrak{A}_k(E)} \|Q\|_{\mathfrak{A}_l(E)}.$$

All the examples mentioned in Example 4.1.6 are multiplicative with constants as in (4.1.2) and the finite type polynomials are dense in each one.

Next we show that multiplicativity passes to the seminorms p_s^x in a similar way.

Remark 6.2.7. Suppose that $\mathfrak{A}$ is multiplicative with constants $c_{k,l}$ as in (4.1.2), and E is a Banach space. Then, each seminorm p_s^x is "almost" multiplicative in the following sense. Given $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, s) > 1$ such that

$$p_s^x(fg) \leq c p_s^x(g) p_{s+\varepsilon}^x(f). \tag{6.2.1}$$

We will prove it for $x = 0$. The general case follows by translation and dilatation. Let $f = \sum_k P_k$ and $g = \sum_k Q_k$ be functions in $H_{b\mathfrak{A}}(B_E)$. Since $\frac{d^n fg(0)}{n!} = \sum_{k=0}^n P_k Q_{n-k}$ and $\mathfrak{A}$ is multiplicative, $\frac{d^n fg(0)}{n!}$ belongs to $\mathfrak{A}_n(E)$. Applying (4.1.4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} s^n \left\| \frac{d^n fg(0)}{n!} \right\|_{\mathfrak{A}_n(E)} &\leq e^2 \sum_{n=0}^{\infty} s^n \sum_{k=0}^{n} \Big(\frac{k(n-k)}{n} \Big)^{1/2} \|P_k\|_{\mathfrak{A}_k(E)} \|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)} \\
&= e^2 \sum_{k=0}^{\infty} \sqrt{k} s^k \|P_k\|_{\mathfrak{A}_k(E)} \sum_{n=k}^{\infty} s^{n-k} \Big(\frac{n-k}{n} \Big)^{1/2} \|Q_{n-k}\|_{\mathfrak{A}_{n-k}(E)} \\
&\leq e^2 p_s^0(g) \sum_{k=0}^{\infty} \sqrt{k} s^k \|P_k\|_{\mathfrak{A}_k(E)}.
\end{aligned}$$

Therefore, for each $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, s) > 1$ such that

$$p_s^0(fg) = \sum s^n \Big\| \frac{d^n fg(0)}{n!} \Big\|_{\mathfrak{A}_n(E)} \leq c p_s^0(g) p_{s+\varepsilon}^0(f).$$

Finally we state Bohr's inequality for analytic functions defined on $\mathbb{C}$.

Theorem 6.2.8. [Bohr's inequality] For every analytic function in $H(\mathbb{C})$, $f(z) = \sum_{n \geq 0} a_n z^n$, the following inequality holds

$$\sup_{|z| \leq 1/3} \sum_{n \geq 0} |a_n z^n| \leq \sup_{|z| \leq 1} \left| \sum_{n \geq 0} a_n z^n \right|.$$

Now we can state and prove Runge's approximation theorem for $\mathcal{H}_{b\mathfrak{A}}(E)$.

Theorem 6.2.9. Let $\mathfrak{A} = \{\mathfrak{A}_k\}_k$ be a holomorphy type, E be a Banach space, $x \in E$, and r, s and δ be positive real numbers. Suppose that finite type polynomials are dense in each $\mathfrak{A}_k$ and that $\mathfrak{A}$ is multiplicative with constants as in (4.1.2). If f is a holomorphic function of $\mathfrak{A}$-bounded type on the disjoint balls $B(0, r + \delta)$ and $B(a, s + \delta)$, then there are polynomials in $\mathcal{H}_{b\mathfrak{A}}(E)$ which approximate f in $\mathcal{H}_{b\mathfrak{A}}(B(0, r/3))$ and $\mathcal{H}_{b\mathfrak{A}}(B(a, s/3))$.

Proof. Let $\varepsilon > 0$. First, since f is holomorphic of $\mathfrak{A}$-bounded type in $B(0, r)$ and $B(a, s)$, there exist polynomials P_1 and P_2 in $\mathcal{H}_{b\mathfrak{A}}(E)$ such that $p_{r/3}^0(P_1 - f) < \varepsilon/3$ and that $p_{s/3}^a(P_2 - f) < \varepsilon/3$.

Let $M = c(p_{r/3+1}^0(P_1) + p_{r/3+1}^0(P_2) + p_{s/3+1}^a(P_1) + p_{s/3+1}^a(P_2))$, where c is a positive constant such that $p_{r/3}^0(gh) \leq c p_{r/3+1}^0(g) p_{r/3}^0(h)$ and $p_{s/3}^a(gh) \leq c p_{s/3+1}^a(g) p_{s/3}^a(h)$ for any polynomials g, h in $\mathcal{H}_{b\mathfrak{A}}(E)$ (see Remark 6.2.7).

By the Hahn-Banach separation theorem, there exists $\varphi \in E'$ such that $K_1 = \overline{\varphi(B(0,r))}$ and $K_2 = \overline{\varphi(B(a,s))}$ are disjoint convex compact sets in $\mathbb{C}$. In fact, since K_1 and K_2 are closed, convex, balanced sets in $\mathbb{C}$, we get that $K_1 = \overline{D}(0, r\|\varphi\|)$ and $K_2 = \overline{D}(\varphi(a), s\|\varphi\|)$. Now we can apply Runge's Theorem to the compact set $K = K_1 \cup K_2 \subset \mathbb{C}$, since $\mathbb{C} \cap K$ is path-connected, and find a polynomial $q \in \mathbb{C}[z]$ such that $|q(z) - 1| < \frac{\varepsilon}{3M}$ for every $z \in K_1$ and $|q(z)| < \frac{\varepsilon}{3M}$ for every $z \in K_2$.

Consider $h = q \circ \varphi \in \mathcal{H}_{b\mathfrak{A}}(E)$. Suppose that $q(z) = \sum_{j=0}^{m} a_j z^j$, then applying Bohr's inequality in $\mathbb{C}$, we get that

$$p^0_{r/3}(h-1) = |a_0 - 1| + \sum_{j=1}^{m} \left(\frac{r}{3}\right)^j |a_j| \|\varphi\|^j \leq \sup_{z \in \overline{D}(0, r\|\varphi\|)} |q(z) - 1| \leq \frac{\varepsilon}{3M}.$$

Suppose also that $q(z) = \sum_{j=0}^{m} b_j (z - \varphi(a))^j$. Applying Bohr's inequality again we get that

$$p^a_{s/3}(h) = \sum_{j=0}^{m} \left(\frac{s}{3}\right)^j |b_j| \|\varphi\|^j \leq \sup_{z \in \overline{D}(\varphi(a), s\|\varphi\|)} |q(z)| \leq \frac{\varepsilon}{3M}.$$

Finally, define $P = P_1 h + P_2(1-h) \in \mathcal{H}_{b\mathfrak{A}}(E)$, thus

$$\begin{aligned} p^0_{r/3}(P - f) &\leq p^0_{r/3}(P_1(h-1)) + p^0_{r/3}(P_1 - f) + p^0_{r/3}(P_2(1-h)) \\ &< c p^0_{r/3+1}(P_1) p^0_{r/3}(h-1) + \frac{\varepsilon}{3} + c p^0_{r/3+1}(P_2) p^0_{r/3}(1-h) \\ &\leq \varepsilon, \end{aligned}$$

$$\begin{aligned} p^a_{s/3}(P - f) &\leq p^a_{s/3}(P_1 h) + p^a_{s/3}(P_2 - f) + p^a_{s/3}(P_2 h)) \\ &< c p^a_{s/3+1}(P_1) p^a_{s/3}(h) + \frac{\varepsilon}{3} + c p^a_{s/3+1}(P_2) p^a_{s/3}(h) \\ &\leq \varepsilon. \end{aligned}$$

□

Denote by $\phi(z) = \lambda z + b$ for $z \in E$ and $\phi_i(z) = \lambda_i z + b_i$ for $z \in \mathbb{C}$. In the next Lemma we will prove case (b) of Theorem 6.2.1, which is the case that one coordinate of the map ϕ is a translation.

Lemma 6.2.10. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s \in \mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (4.1.2), such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Let $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ be defined by $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, and suppose that $|\lambda^\alpha| < 1$, $\|\lambda\|_\infty = 1$ and there exist some coordinate with $\lambda_k = 1$ and $b_k \neq 0$. Then T is a mixing operator.

Proof. We want to show that T is a mixing operator, i.e, for every pair of open sets U and V in $\mathcal{H}_{b\mathfrak{A}}(E)$, there exist a positive integer n_0 for which $T^n U \cap V \neq \emptyset$, for all $n \geq n_0$. Without loss of generality we can suppose that

$$U = \{h \in \mathcal{H}_{b\mathfrak{A}}(E) \text{ such that } p^0_r(h - f) < \delta\} \quad \text{and} \quad V = \{h \in \mathcal{H}_{b\mathfrak{A}}(E) \text{ such that } p^0_r(h - g) < \delta\},$$

for f, $g \in \mathcal{H}_{b\mathfrak{A}}(E)$ and r, δ positive numbers. Since, E has a shrinking basis, by Lemma 6.1.6, we can assume that f is a finite linear combination of monomials. Define an inverse for T over the span of the monomials by integrating each monomial and denote by S.

Applying (6.1.3) several times, each time dividing $\varepsilon = 1$ by 2 we get that for all $x \in E$

$$p_r^x(T^n f) \leq C(n,\alpha) p_{r+1}^{\phi^n(x)}(f).$$

Thus,

$$p_r^0(T^n q - f) = p_r^0(T^n(q - S^n f)) \leq C(n,\alpha) p_{r+1}^{\phi^n(0)}(q - S^n f).$$

The fact that $\lambda_k = 1$ and $b_k \neq 0$, implies that $(\phi^n(0))_k = n b_k$. Since E has a 1-unconditional basis we get that,

$$\|x - \phi^n(0)\|_E \geq n|b_k|.$$

If $\|x - \phi^n(0)\| \geq 6r + 5$, we get that $B(0, 3r+1) \cap B(\phi^n(0), 3(r+1)+1) = \emptyset$. Then, by Theorem 6.2.9, there exist a polynomial $q \in \mathcal{H}_{b\mathfrak{A}}(E)$, such that

$$p_r^0(q - g) < \delta \ \text{ and } \ p_{r+1}^{\phi^n(0)}(q - S^n f) < \frac{\delta}{C(n,\alpha)}.$$

Then, we get that for all $n \in \mathbb{N}$ such that $n|b_k| > 6r + 5$, there exist a polynomial $q \in \mathcal{H}_{b\mathfrak{A}}(E)$, such that

$$p_r^0(q - g) < \delta \ \text{ and } \ p_r^0(T^n q - f) < \delta.$$

So, we have prove that there is a positive integer n_0 for which $T^n U \cap V \neq \emptyset$, for all $n \geq n_0$. $\square$

Now will we take care of the cases (c) and (d) of our main theorem, in which no coordinate of the function ϕ is a translation. Note that the fact that $\phi_i(z) = \lambda_i z + b_i$ is not a translation implies that ϕ_i has a fixed point on $b_i/(1 - \lambda_i)$ (here we suppose that 0/0=0).

Since the basis of E, $(e_s)_{s\in\mathbb{N}}$, is shrinking, Proposition 1.b.2 of [JL79] implies that the bidual of E can be identified with those sequence of complex numbers $(z_1, z_2, z_3, \dots)$ such that

$$\sup_n \left\| \sum_{i=1}^n z_i e_i \right\| < \infty.$$

This correspondence is given by

$$z'' \leftrightarrow (z''(e_1'), z''(e_2'), z''(e_3'), \dots),$$

and the norm of z'' is equivalent to $\sup_n \| \sum_{i=1}^n z''(e_i') e_i \|$.

Let us denote by $\zeta = (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \in \mathbb{C}^{\mathbb{N}}$ the sequence of the fixed points of each ϕ_i. We are going to consider the cases in which $\zeta \in E''$ and $\zeta \notin E''$. We start with the case $\zeta \in E''$.

Lemma 6.2.11. Let E be a Banach space with a shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $X \subset H_b(E)$ be a Fréchet space of holomorphic functions of bounded type. Suppose that $T : X \to X$ is defined by $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, with $|\lambda^\alpha| < 1$ and $\zeta \in E''$. Then T is not hypercyclic.

Proof. Let us denote the Aron-Berner extension defined over $H_b(E)$ by $AB : H_b(E) \to H_b(E'')$, $AB(\sum_k P_k) = \sum_k AB(P_k)$. Also denote by q_r the seminorms on $H_b(E)$,

$$q_r(\sum_k P_k) = \sum_{k\geq 0} r^k \|P_k\|_{\mathcal{P}^k(E)}.$$

Recall that AB is a continuous map

$$q_r\left(AB\left(\sum_k (P_k)\right)\right) = \sum_{k\geq 0} r^k \|AB(P_k)\|_{\mathcal{P}^k(E'')} \leq \sum_{k\geq 0} r^k \|P_k\|_{\mathcal{P}^k(E)} = q_r\left(\sum_k P_k\right).$$

Finally, denote the evaluation at ζ by $\delta_\zeta : H_b(E'') \to \mathbb{C}$, $\delta_\zeta(g) = g(\zeta)$.
If $g = \sum_k P_k \in H_b(E'')$, we get that

$$|g(\zeta)| = |\sum_{k\geq 0} P_k(\zeta)| \leq \sum_{k\geq 0} \|\zeta\|_{E''}^k \|P_k\|_{\mathcal{P}^k(E'')} = q_{\|\zeta\|}(g).$$

Under this assumptions, we can prove that no orbit of T can be dense. First recall that every orbit of T has the following form:

$$T^n f(z) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\phi^n(z)\right) = \lambda^{\frac{n(n-1)}{2}\alpha} D^{n\alpha} f\left(\lambda^n z + b\frac{1-\lambda^n}{1-\lambda}\right).$$

Thus, since ϕ^n is an affine map, we get that

$$\delta_\zeta(AB(\lambda^n z + b\frac{1-\lambda^n}{1-\lambda})) = \zeta.$$

Now, we are able to show that T is not hypercyclic,

$$\begin{aligned}
|\delta_\zeta AB(T^n f)| &= |\lambda^\alpha|^{\frac{n(n-1)}{2}} |\delta_\zeta AB(D^{n\alpha} f \circ \phi^n)| = |\lambda^\alpha|^{\frac{n(n-1)}{2}} |\delta_\zeta[AB(D^{n\alpha} f) \circ AB(\phi^n)]| \\
&= |\lambda^\alpha|^{\frac{n(n-1)}{2}} |AB(D^{n\alpha} f)(AB(\phi^n)(\zeta))| = |\lambda^\alpha|^{\frac{n(n-1)}{2}} |AB(D^{n\alpha} f)(\zeta)| \\
&\leq |\lambda^\alpha|^{\frac{n(n-1)}{2}} q_{\|\zeta\|}(D^{n\alpha} f) \\
&\leq |\lambda^\alpha|^{\frac{n(n-1)}{2}} e^{n|\alpha|+1} n^{|\alpha|/2} \left(\prod_{\alpha_i \neq 0} \alpha_i\right)^{1/2} q_{\|\zeta\|+1}(f) \underset{n\to\infty}{\to} 0.
\end{aligned}$$

Since, $\delta_\zeta \circ AB$ is a surjective continuous map, then no orbit of T can be dense in $H_b(E)$. Thus, T is not hypercyclic. □

The last case it remains to be shown is when $|\lambda^\alpha| < 1$ and $\zeta \notin E''$. We will restrict ourselves to the case $\|\lambda\|_\infty = 1$. Note that if $\|\lambda\|_\infty < 1$, then $\zeta \in E$, thus T is not hypercyclic. If $\|\lambda\|_\infty > 1$, then the inequality (6.1.3) for the operator in $\mathcal{H}_{b\mathfrak{A}}(E)$ is not useful for us, because we are not able to prove that ϕ is runaway. If we restrict to $\|\lambda\|_\infty = 1$, we can prove that the operator is mixing in $\mathcal{H}_{b\mathfrak{A}}(E)$. Furthermore, if $\mathfrak{A}$ is the sequence of ideals of approximable polynomials, then we can dispense the condition on $\|\lambda\|_\infty$ in the space $H_{bc}(E)$ of entire functions of compact bounded type, as we will see at the end of this section.

Lemma 6.2.12. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let $\mathfrak{A}$ be a multiplicative holomorphy type with constants as in (4.1.2), such that the finite type polynomials are dense in each $\mathfrak{A}_k(E)$. Let $T : \mathcal{H}_{b\mathfrak{A}}(E) \to \mathcal{H}_{b\mathfrak{A}}(E)$ be defined by $Tf = M_\lambda \circ \tau_b \circ D^\alpha(f)$, and suppose that $|\lambda^\alpha| < 1$, $\|\lambda\|_\infty = 1$ and that $\zeta \notin E''$. Then T is a mixing operator.

Proof. We want to prove that T is a mixing operator. Just like in the proof of Lemma 6.2.10, we fix a pair of open sets U and V in $\mathcal{H}_{b\mathfrak{A}}(E)$. We will show the existence of a positive integer k_0 for which $T^kU \cap V \neq \emptyset$, for all $k \geq k_0$. Without loss of generality we can suppose that

$$U = \{h \in \mathcal{H}_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h-f) < \delta\} \quad \text{and} \quad V = \{h \in \mathcal{H}_{b\mathfrak{A}}(E) \text{ such that } p_r^0(h-g) < \delta\},$$

for f, $g \in \mathcal{H}_{b\mathfrak{A}}(E)$ and r, δ positive numbers. Since, E has a shrinking basis, by Lemma 6.1.6, we can assume that f is a finite linear combination of monomials. Define an inverse for T over the span of the monomials by integrating each monomial and denote it by S.

Applying (6.1.3) several times, each time dividing $\varepsilon = 1$ by 2 we get that for all $x \in E$

$$p_r^x(T^k f) \leq C(k,\alpha) p_{r+1}^{\phi^k(x)}(f).$$

Thus,

$$p_r^0(T^k q - f) = p_r^0(T^k(q - S^k f)) \leq C(k,\alpha) p_{r+1}^{\phi^k(0)}(q - S^k f).$$

It is enough to show that the sequence $\phi^k(0)$ is not bounded, because in that case, there exists some $k_0 \in \mathbb{N}$ such that the balls $B(0,r)$ and $B(\phi^k(0), r+1)$ are disjoint for all $k \geq k_0$. By an application of Theorem 6.2.9, it follows that $T^kU \cap V \neq \emptyset$ for all $k \geq k_0$.

A simple calculation shows that

$$\phi^k(0) = \sum_{j\in\mathbb{N}} b_j \frac{\lambda_j^k - 1}{\lambda_j - 1} e_j.$$

Note that we can decompose $\mathbb{N} = N_1 \cup N_2$, in two disjoint subsets with

$$N_1 = \{n \in \mathbb{N} : |\lambda_n| = 1\} \text{ and } N_2 = \{n \in \mathbb{N} : |\lambda_n| < 1\}.$$

Define then for i, $i = 1, 2$ the vector ζ^i with $\zeta^i_n = \zeta_n$ for $n \in N_i$ and $\zeta^i_n = 0$ for $n \notin N_i$. Note that $\zeta = \zeta^1 + \zeta^2$.

We will divide the proof in two cases. First we will prove that the sequence $\phi^k(0)$ is not bounded if $\zeta^1 \notin E''$, and then we will do so if $\zeta^2 \notin E''$.

Suppose first that $\zeta^1 \notin E''$. Denote by $|||z||| = \sup_k \left\| \sum_{i=1}^k z_i e_i \right\|$, which is an equivalent norm in E. Suppose that there exist some positive constant C such that $\left|\left|\left|\phi^k(0)\right|\right|\right| \leq C$ for all $k \in \mathbb{N}$. Then we get that

$$\frac{1}{N} \sum_{j=1}^N |||\phi^j(0)||| \leq C.$$

We will show that this leads to a contradiction. Since $\zeta^1 \notin E''$, let $A \in \mathbb{N}$ be a finite subset on N_1 such that $\lambda_n \neq 1$ if $n \in A$ and such that

$$\left\| \sum_{l\in A} \frac{b_l}{\lambda_l - 1} e_l \right\| \geq 2C.$$

Since E has a 1-unconditional basis, we get that

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^{N}|||\phi^j(0)||| \geq \frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{l\in A}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| &\geq \frac{1}{N}\sum_{j=1}^{N}\left\|\sum_{l\in A}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &\geq \left\|\frac{1}{N}\sum_{j=1}^{N}\sum_{l\in A}(\lambda_l^j-1)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &= \left\|\sum_{l\in A}\frac{b_l}{\lambda_l-1}e_l\left[\frac{1}{N}\sum_{j=1}^{N}(\lambda_l^j-1)\right]\right\|\end{aligned}$$

Since $|\lambda_l|=1$ and $\lambda_l\neq 1$ for all $l\in N_1$, we can write $\lambda_l=e^{i\rho_l}$. Thus, if $l\in A$, we get that

$$\begin{aligned}\frac{1}{N}\left|\sum_{j=1}^{N}(\lambda_l^j-1)\right| = \left|\left(\frac{1}{N}\sum_{j=1}^{N}\lambda_l^j\right)-1\right| &= \left|\frac{1}{N}\frac{e^{i(N+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}-1\right| \\ &\geq 1-\frac{1}{N}\left|\frac{e^{i(N+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}\right|\end{aligned}$$

Now, given $\eta>0$, we can fix $K\in\mathbb{N}$ such that

$$\frac{1}{K}\left|\frac{e^{i(K+1)\rho_l}-e^{2i\rho_l}}{e^{i\rho_l}-1}\right| \leq \frac{2}{K\min_{l\in A}|e^{i\rho_l}-1|}\leq\eta.$$

Finally, we get that for $l\in A$

$$\frac{1}{K}\left|\sum_{j=1}^{K}(\lambda_l^j-1)\right|\geq 1-\eta,$$

which means that

$$\begin{aligned}\frac{1}{K}\sum_{j=1}^{K}|||\phi^j(0)||| &\geq \left\|\sum_{l\in A}\frac{b_l}{\lambda_l-1}e_l\left[\frac{1}{K}\sum_{j=1}^{K}(\lambda_l^j-1)\right]\right\| \\ &\geq \left\|\sum_{l\in A}(1-\eta)\frac{b_l}{\lambda_l-1}e_l\right\| \\ &> (1-\eta)2C.\end{aligned}$$

It follows that the sequence $\phi^k(0)$ is not bounded.

Now we assume that $\zeta_2\notin E''$. If $j\in N_2$, we have that $|\lambda_j|<1$, which implies that

$$\lim_{k\to\infty}\phi^k(0)_j=\lim_{k\to\infty}b_j\frac{\lambda_j^k-1}{\lambda_j-1}=\frac{b_j}{1-\lambda_j}=\zeta_j^2.$$

Suppose that $\phi^k(0)$ is bounded. It follows that $\phi^k(0)$ has a w^*-accumulation point $z\in E''$ and that

$$\lim_{k\to\infty}\phi^k(0)_j=z_j.$$

Thus, $z_j=\zeta_j^2$ for all $j\in N_2$. It follows that $\zeta^2\in E''$, which is a contradiction. This proves that the sequence $\phi^k(0)$ is not bounded, hence the operator T is mixing as we wanted to prove. □

6.2.1 Holomorphic functions of compact bounded type

In this section we deal with the case in which $\mathfrak{A} = \mathcal{A}$, is the sequence of ideals of approximable polynomials. Then $H_{b\mathcal{A}}(E)$ is the space $H_{bc}(E)$ of complex valued, entire functions on E of compact type that are bounded on bounded subsets of E. We take special interest in this case for it's similarities whit the case of holomorphic functions on finite variables, which we already study in [MPS14]. The space $H_{bc}(E)$ is endowed with the topology of uniform convergence on balls of E. Hence, we consider the following family of seminorms that generates the topology of this space. Given a bounded set $A \subset E$ and $f \in H_{bc}(E)$, we define

$$p_A(f) = \sup_{z \in A} |f(z)|.$$

Since this topology is the same as the one for holomorphic functions on finite complex variables, we will show that we can dispense the assumption on $\|\lambda\|_\infty$ just as in the finite variables setting. It is clear that statements (a) and (d) of Theorem 6.2.1, remain the same. Our objective is to prove the following more general versions of the statements (b) and (c) of our main theorem. As we mentioned previously $\mathcal{A}$ is a multiplicative holomorphy type in which the finite type polynomials are dense in each $\mathcal{A}_k(E)$ and AB-closed. In this particular context our result reads as follows.

Theorem 6.2.13. Let E be a Banach space with a 1-unconditional shrinking basis, $(e_s)_{s\in\mathbb{N}}$. Let T be the operator on $H_{bc}(E)$, defined by $Tf(z) = M_\lambda \circ \tau_b \circ D^\alpha f(z)$, with $\alpha \neq 0$ and $\lambda_i \neq 0$ for all $i \in \mathbb{N}$. Then,

a) If $|\lambda^\alpha| \geq 1$ then T is strongly mixing in the gaussian sense.

b) If for some $i \in \mathbb{N}$ we have that $b_i \neq 0$ and $\lambda_i = 1$, then T is mixing.

c) If $\zeta := (b_1/(1-\lambda_1), b_2/(1-\lambda_2), b_3/(1-\lambda_3), \dots) \notin E''$, then T is mixing.

d) If $\zeta \in E''$, then T is not hypercyclic.

The key point to prove this new statements is that under this assumptions the affine symbol ϕ will result to be runaway. Then, applying Theorem 6.2.9 we will be able to prove that the operator is mixing. During this section E will denote a Banach space with separable dual and suppose that $(e_s)_{s\in\mathbb{N}}$ is a 1-unconditional shrinking basis. In order to prove that the operator T is mixing on $H_{bc}(E)$ we need to give bounds for $p_A(D^\alpha f)$ in terms of $p_A(f)$, eventually by enlarging if necessary the set A. For this we will assume that the space E is of the form $\mathbb{C}^N \times F$, and that α only have nonzero coordinates in corresponding to the coordinates of $\mathbb{C}^N$.

Remark 6.2.14. Let $A = A_1 \times A'$ be a bounded subset of $E = \mathbb{C}^N \times F$ and suppose that $\alpha_i = 0$ for every $i > N$. If $f \in H_{bc}(E)$ and $z = (z_1, \dots, z_N, z') \in E$, then

$$D^\alpha f(z_1, \dots, z_N, z') = \frac{\alpha!}{(2\pi i)^N} \int_{|w_1 - z_1| = r_1} \cdots \int_{|w_N - z_N| = r_N} \frac{f(w_1, \dots, w_N, z')}{\prod_{i=1}^N (w_i - z_i)^{\alpha_i + 1}} \, dw_1 \dots dw_N.$$

Therefore, we can estimate the seminorm of $D^\alpha f$ over $A = B(z_1, r_1) \times \cdots \times B(z_N, r_N) \times A'$, where $B(z_j, r_j)$ denotes the closed disk of center $z_j \in \mathbb{C}$ and radius r_j. Fix positive real numbers $\varepsilon_1, \dots, \varepsilon_N$, then

$$p_A(D^\alpha f) \leq \frac{\alpha!}{(2\pi)^N} \frac{p_{(A_1 + \varepsilon, A')}(f)}{\varepsilon_1^{\alpha_1 + 1} \cdots \varepsilon_N^{\alpha_N + 1}}. \tag{6.2.2}$$

The case *b*) follows the lines of the case *b*) of [MPSc, Theorem 3.4]. Actually the same proof remains valid adapting the bounded sets to this case. To prove the case *c*) we proceed in a similar way to the proof of it counterpart on Theorem 6.2.1. We can decompose the hole space E in two subspaces corresponding to the different sizes of the modulus of λ_i. Decompose $\mathbb{N} = N_1 \cup N_2$, into two disjoint subsets with

$$N_1 = \{n \in \mathbb{N} : |\lambda_n| \leq 1\} \text{ and } N_2 = \{n \in \mathbb{N} : |\lambda_n| > 1\}.$$

We have that $E = E(N_1) + E(N_2)$. Define for i, $i = 1, 2$ the vector ζ^i with $\zeta^i_n = \zeta_n$ for $n \in N_i$ and $\zeta^i_n = 0$ for $n \notin N_i$. Note that $\zeta = \zeta^1 + \zeta^2$. If $\zeta^1 \notin E''$, then following the lines of the proof of part *c*) in Theorem 6.2.1, we can conclude that ϕ is runaway, so that the operator $C_\phi \circ D^\alpha$ is mixing. Otherwise, if $\zeta^2 \notin E''$ and since $|\lambda_i| > 1$ for every $i \in N_2$, we can consider $\phi_2^{-1} : E(N_2) \to E(N_2)$. It is easy to see that ζ^2 is the fixed point of ϕ_2 and that $\phi_2(z) = \frac{1}{\lambda}(z - b)$. Since, $|\lambda_i| > 1$ for every $i \in N_2$, we get again by following the proof of part *c*) Theorem 6.2.1 that ϕ_2 is runaway. Now, since the topology on $H_{bc}(E)$ is the topology of uniform convergence on bounded sets, we get that ϕ is runaway and thus $C_\phi \circ D^\alpha$ is mixing by Theorem 6.2.9.

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